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Finite-Difference Approximations And Optimal Control Of Differential Inclusions

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FINITE-DIFFERENCE APPROXIMATIONS AND OPTIMAL CONTROL OF DIFFERENTIAL INCLUSIONS

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To my parents

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Chapter 1

Introduction

Optimal control is among the most important motivations and applications of modern methods os variational analysis, which concerns the properties of control functions that give solution which minimize a cost function, when inserted into a differential equation. The differential inclusion model encompasses ordinary differential equations control systems represented in the parameterized control form, which make optimal control systems described by ordinary differential equations can be generalized to optimization problems governed by differential inclusions. However, this kind of optimization problem is intrinsically nonsmooth, (even the cost function and all the real-valued functions in the problem is smooth) due to set-valued dynamic constraints. Thus the usage and development of appropriate tools of variational analysis and generalized differentiation are required for the study and applications of (P) and related problems governed by differential inclusions. The method of discrete approximations has been well recognized as an efficient approach to investigate differential inclusions and optimization problems for them from both qualitative and quantitative/numerical viewpoints; see, e.g., the surveys and books [16, 22, 30, 34] and the bibliographies therein.

This dissertation addresses the following optimization problem (P) of the *generalized Bolza type* for dynamic systems governed by constrained differential inclusions with general initial conditions and endpoint constraints:

$$\text{minimize } J[x] := \varphi\big(x(T)\big) + \int_0^T f\big(x(t), \dot{x}(t), t\big)dt \tag{1.1}$$

over absolutely continuous trajectories $x : [0,T] \to \mathbb{R}^n$ satisfying the differential inclusion

$$\dot{x}(t) \in F\big(x(t),t\big) \text{ a.e. } t \in [0,T] \text{ with } x(0) = x_0 \in \mathbb{R}^n \tag{1.2}$$

subject to geometric endpoint constraints,

$$x(T) \in \Omega \subset \mathbb{R}^n. \tag{1.3}$$

Here x_0 is a fixed n-vector, $F\colon \mathbb{R}^n \times [0,T] \rightrightarrows \mathbb{R}^n$ is a set-valued mapping/multifunction, Ω is an nonempty set, f and φ are real-valued functions.

Observe that the differential inclusion framework (1.2) covers not only standard control systems with constant control sets but also significantly more challenging problems with *feedback* reflected by the dependence of the control sets in ordinary differential equations on state variables. Differential inclusion problems of type (P) have been well recognized in dynamic optimization and control theory as a convenient framework to cover the vast majority of conventional and nonconventional models arising in optimization and control of dynamical systems described via time derivatives. We refer the reader to the books [30, 38] and the bibliographies therein for more discussions, historical overviews as well as applied models governed by differential inclusion.

For the above Bolza problem, the *method of discrete approximations* allows us to approximate this continuous-time problem by those involving discrete dynamics. A principal question arising in all the aspects and modifications of this method (even without applications to optimization) is about the possibility to approximate, in a suitable sense, feasible trajectories of the given differential inclusion by those for finite-difference inclusions that appear by using one or another scheme to replace time derivatives. The majority of the results in this direction concern explicit Euler schemes under the Lipschitz continuity of velocity mappings with respect to state variables; see [2, 16, 22, 27, 30, 34] for more details and references.

Another line of numerical method for differential inclusions is *higher-order discrete approximation*; see, e.g., [1, 21, 23, 35, 36, 37]. The previous work almost focus on the convergence rates and the error estimate for discrete approximations. The second-order approximationis was given in the case if the right hand side set value mapping F is strongly convex valued and Lipschitz continuous in [35]. Veliov showed that for the general form of *second-order Runge-Kutta schemes*, if the Lipschitz condition is not satisfied, there is barely hope that we get higher-order convergence. Some second-order discrete approximations to particular classes of differential inclusion was introduced by Veliov in [35, 36, 37].

The other lines of research on discrete approximations of differential inclusions via the explicit Euler schemes invoke the replacement of the Lipschitz continuity of velocity mappings by various *one-sided Lipschitzian* conditions; see, e.g., [8, 9, 10, 11, 22]. Conditions of this type essentially weaken, from one side, the classical Lipschitz continuity, while from the other side they encompass *dissipativity* properties widely used in nonlinear analysis and the theory of monotone operators. Note to this end the so-called *modified one-sided Lipschitzian* (MOSL) condition introduced and applied in [11] to justify a certain strong approximation of solution sets for differential inclusions by finite-difference ones obtained via the explicit Euler scheme and to derive in this way a Bogolyubov-type density theorem for the Bolza problem (P) and the corresponding convergence of discrete optimal solutions.

The first part of the dissertation is devoted to implicit Euler approximation and optimization of one-sided Lipschitzian differential inclusions. In this chapter we consider more generalized problem $(\widehat{P})$ that is the Bolza problem (P) with functional endpoint constraints of the inequality and equality types given by

$$\varphi_i(x(T)) \leq 0 \ \text{ for } \ i = 1, \ldots, m, \tag{1.4}$$

$$\varphi_i(x(T)) = 0 \ \text{ for } \ i = m+1, \ldots, m+r. \tag{1.5}$$

First, We construct discrete approximations of differential inclusions with relaxed one-sided Lipschitzian (ROSL) right-hand sides by using the implicit Euler scheme for approximating time derivatives, and then we justify an appropriate well-posedness of such approximations. Our principal result establishes the uniform approximation of strong local minimizers for the continuous-time Bolza problem by optimal solutions to the implicitly discretized finite-difference systems in the general ROSL setting and even by the strengthen $W^{1,2}$-norm approximation of this type in the case intermediate local minimizers under additional assumptions. Finally, we derive necessary optimality conditions for the discretized Bolza problems via suitable generalized differential constructions of variational analysis.

The second part of the dissertation focus on Runge-Kutta approximation and optimization of one-sided Lipschitzian differential inclusions. In this chapter, instead of evaluating the error estimate, we construct an approximating Runge-Kutta sequence and prove that this sequence converges to the optimal solution. In this chapter we study the generalized problem (P) which only with the geometry endpoints constraints. First we establish well-posedness of the Runge-Kutta discrete approximations in the sense of $W^{1,2}$-norm convergence to the trajectory for differential inclusions. The Runge-Kutta discrete approximations allows us to build a well-posed sequence of finite-dimensional optimization problems with a strong convergence of optimal solution. Based on the advanced tools of variational analysis and generalized differentiation, we derive necessary optimality conditions for discrete-time problems.

The obtained results on the well-posedness of discrete approximations and necessary optimality conditions allow us to justify a numerical approach to solve the generalized Bolza problem for differential inclusions by using discrete approximations constructed via the implicit Euler scheme and the Runge-Kutta scheme.

In Chapter 2 we introduce some preliminaries from variational analysis and generalized differentiation and present some of their properties which are appropriate for the main objective of this dissertation. The notions included one-sided Lipschitzian condition, normal cones, subdifferentials, coderivatives, and the review of their important properties. Some preliminaries, like the solvability of the implicit Euler scheme and necessary conditions for mathematical programs, used in the proofs of our results. The appropriate tools of generalized differentiation are introduced by Mordukhovich [29, 30] and then developed and applied in many publications.

Besides Chapter 2, this main result of this thesis contains are presented in Chapter 3 and Chapter 4. In Chapter 3, We construct discrete approximations of differential inclusions with relaxed one-sided Lipschitzian (ROSL) right-hand sides by using the implicit Euler scheme for approximating time derivatives, and then we justify an appropriate well-posedness of such approximations. Our principal result establishes the uniform approximation of strong local minimizers for the continuous-time Bolza problem by optimal solutions to the implicitly discretized finite-difference systems in the general ROSL setting and even by the strengthen $W^{1,2}$-norm approximation of this type in the case intermediate local minimizers under additional assumptions. Finally, we derive necessary optimality conditions for the discretized Bolza problems via suitable generalized differential constructions of variational analysis and then for the original

Bolza problem by passing to the limit.

Chapter 4 focus on the Runge-Kutta discrete approximations for the differential inclusion. We justify the well-posedness of such approximation. We construct well-posed discrete approximations of the original continuous-time Bolza problem (P) and establish their strong convergence to the intermediate local minimizers of (P). Based on the generalized differentiation, necessary optimality conditions are obtained for the discrete approximation problems under additional assumption. Finally, in Chapter 5, we introduce some issues we are working further.

Chapter 2

Preliminaries

In this chapter we present some basic definitions and preliminary materials of variational analysis and generalized differentiation, which are widely used in the formulations and proof of the major results. We refer the reader to [29, 33] for more details, discussions and the extensive bibliography. Recall that $\mathbb{R}^n$ denotes the n-dimensional space with the Euclidean norm $|\cdot|$ and the closed unit ball $I\!\!B$ and that $\mathcal{CC}(\mathbb{R}^n)$ signifies the space of convex and compact subsets of $\mathbb{R}^n$ endowed with the Pompieu-Hausdorff metric. For a non empty subset $\Omega \subset \mathbb{R}^n$, the expressions $\mathrm{cl}\,\Omega$, $\mathrm{co}\,\Omega$, $\mathrm{clco}\,\Omega$ stand for the standard notation of closure, convex hull, closed convex hull, respectively. The distance function associated with an nonempty closed set $\Omega \subset \mathbb{R}^n$ is denoted by

$$\mathrm{dist}(x, \Omega) := \min_{y \in \Omega} |x - y|, \quad x \in \mathbb{R}^n,$$

and the distance between two closed sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ is given by

$$\mathrm{dist}(\Omega_1, \Omega_2) := \max\Big\{ \max_{x \in \Omega_1} \mathrm{dist}(x, \Omega_2),\ \max_{y \in \Omega_2} \mathrm{dist}(y, \Omega_1) \Big\}. \tag{2.1}$$

$W^{1,p}[0,T]$, $(1 \le p \le \infty)$, is the Sobolev space, in particular $W^{1,2}[0,T]$ is the Sobolev space of absolutely continuous functions $x: [0,T] \to \mathbb{R}^n$ with the norm

$$||x(\cdot)||_{W^{1,2}} := \max_{t \in [0,T]} ||x(t)|| + \Big(\int_0^T ||\dot{x}(t)||^2 \Big)^{\frac{1}{2}},$$

Given an arbitrary a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the Painlevé-Kuratowski outer limit of F as $x \to \bar{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \Big|\ \exists\, x_k \to \bar{x},\ y_k \to y \ \text{ with } \ y_k \in F(x_k),\ k \in \mathbb{N} \Big\}. \tag{2.2}$$

The following property introduced in [8] is our standing assumption on the right-hand side $F(\cdot, t)$ of the differential inclusion in (1.2) playing a crucial role in this dissertation.

A set-valued mapping $F : \mathbb{R}^n \to \mathcal{CC}(\mathbb{R}^n)$ is called to be *relaxed one-sided Lipschitzian (ROSL)* with constant $l \in \mathbb{R}$ if for any given $x_1, x_2 \in \mathbb{R}^n$ and $y_1 \in F(x_1)$ there exits $y_2 \in F(x_2)$ such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle \le l |x_1 - x_2|^2. \tag{2.3}$$

Note that the number/modulus l in (2.3) is not required to be positive as in the classical Lipschitz continuity. The ROSL condition is dramatically weaker the standard Lipschitz continuity and essentially relaxes dissipativity and other one-sided Lipschitzian properties; see more discussions and examples in [8, 9, 10, 22].

The next result on the solvability of the *implicit Euler scheme*

$$\Phi_h(x) := \{ y \in \mathbb{R}^n |\ y \in x + hF(y)\}, \quad h > 0, \tag{2.4}$$

under the ROSL condition is taken from [4, Theorem 4] (the proof of which is based on the Kakutani fixed-point theorem) and is useful in what follows. Recall that a set-valued mapping F is upper semicontinuous (usc) on $\mathbb{R}^n$ if for any $\bar{x} \in \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $\gamma > 0$ such that $F(x) \subset F(\bar{x}) + \varepsilon \mathbb{B}$ whenever $|x - \bar{x}| \le \gamma$.

Lemma 2.1 (solvability of the implicit Euler scheme). *Let $F\colon \mathbb{R}^n \to \mathcal{CC}(\mathbb{R}^n)$ be usc and*

ROSL on $\mathbb{R}^n$ with constraint $l \in \mathbb{R}$ such that $lh < 1$. Then for any $x, y \in \mathbb{R}^n$, there exists a solution $\bar{y} \in \Phi_h(x)$ of the implicit Euler scheme (2.4) *satisfying the estimate*

$$|\bar{y} - y| \leq \frac{1}{1 - lh} \text{dist}(y, x + hF(y)).$$

Following [27], we say that a feasible solution $\bar{x}(\cdot)$ to (P) is an *intermediate local minimizer* (i.l.m.) of rank $p \in [1, \infty)$ for this problem if there are positive numbers ε, α such that $J[\bar{x}] \leq J[x]$ for any other feasible solutions $x(\cdot)$ to (P) satisfying the conditions

$$|x(t) - \bar{x}(t)| < \varepsilon \ \text{ as } \ t \in [0, T] \quad \text{and} \quad \alpha \int_0^T |\dot{x}(t) - \dot{\bar{x}}(t)|^p dt < \varepsilon. \tag{2.5}$$

The case of $\alpha = 0$ in (2.5) corresponds to the classical notion of *strong* local minimum and surely includes global solutions to (P). The notion of *weak* local minimum corresponds to (2.5) with $\alpha \neq 0$ and $p = \infty$; see [27, 30] for detailed discussions and examples.

In what follows we need a certain modification of the i.l.m. notion formulated above, which related to some local relaxation stability of the initial problem (P). Along with (P), consider its extended/relaxed version constructed in the line well understood in the calculus of variations and optimal control. Let

$$f_F(x, v, t) := f(x, v, t) + \delta(v, F(x, t)), \tag{2.6}$$

where $\delta(\cdot, \Lambda)$ is the indicator function of the set Λ equal to 0 on Λ and to ∞ otherwise. Denote by $\widehat{f}_F(x, v, t)$ the convexification for f_F in the v variable, i.e., the largest convex function majorized by $f_F(x, \cdot, t)$ for each x and t. The *relaxed generalized Bolza problem* (R) consists of minimizing the functional

$$\widehat{J}[x] := \varphi(x(T)) + \int_0^T \widehat{f}_F(x(t), \dot{x}(t), t)dt \tag{2.7}$$

over absolutely continuous trajectories $x\colon [0,T] \to \mathbb{R}^n$ under all the endpoint constraints. If $\widehat{J}[x] < \infty$, then $x(\cdot)$ satisfies the convexified differential inclusion

$$\dot{x}(t) \in \text{co}F(x(t), t) \ \text{ a.e. } \ t \in [0,T]. \tag{2.8}$$

Any trajectory for (2.8) is called a relaxed trajectory for (1.2). It is well known that under natural assumptions involving Lipschitzness of F in x, the following approximation property holds: Every relaxed trajectory $x(\cdot)$ can be uniformly approximated in $[0,T]$ by original trajectories $x_k(\cdot)$ starting with the same initial state (but may not satisfy endpoint constraints) such that

$$\liminf \int_0^T f(x_k(t), \dot{x}_k(t), t)\, dt \le \int_0^T \widehat{f}_F(x(t), \dot{x}(t), t)\, dt \text{ as } k \to \infty. \tag{2.9}$$

Note that the relaxed problem (R) reduces to the original one (P) if $F(x,t)$ has convex and compact values and the integrand f is convex with respect to the velocity variable v; in particular, when f does not depend on v. Furthermore, a remarkable fact for the continuous-time problems under consideration consists of the equality between the infimum values of the coast functionals in (P) and (R), without taking endpoint constraints into account, even when f is not convex in v. This fact is known as "hidden convexity" of continuous-valued variational and control problems and relates to the fundamental results of Bogolyubov's and Lyapunov's types; see, e.g., the books [3, 30, 38] for exact formulations and more discussions. The most recent extended version of the Bogolyubov theorem for differential inclusion problems of type (P) was obtained in [11] under the MOSL condition on $F(\cdot, t)$ mentioned in Chapter 1. This discussion makes more natural the following notion taken from [27].

Following [27], we say that a feasible solution $\bar{x}(\cdot)$ to the original problem (P) is called a *relaxed intermediate local minimum* (*r.i.l.m.*) of rank $p \in [1, \infty)$ for (P) if it provides an intermediate local minimum of rank p to the relaxed problem (R) and satisfies the condition $J[\bar{x}] = \widehat{J}[\bar{x}]$.

Moreover, we recall and briefly discuss the generalized differential constructions of variational analysis introduced by the first author [25] and employed in Section 3.4 and 4.4 for deriving necessary optimality conditions; see the books [29, 33] for more details and references on these and related constructions. Given a set $\Omega \subset \mathbb{R}^n$ locally closed around $\bar{x} \in \Omega$, the ε-*normal cone* to Ω at $\bar{x}$ is defined by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \Big\{ v \in \mathbb{R}^n \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le \varepsilon \Big\}. \tag{2.10}$$

When $\varepsilon = 0$, we simply denote $\widehat{N}_0(\bar{x}; \Omega)$ as $\widehat{N}(\bar{x}; \Omega)$ which called *regular normal cone* (known also as the Fréchet normal cone) to Ω at $\bar{x}$. Then the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x}; \Omega) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\Omega} \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega). \tag{2.11}$$

The equivalently defined limiting normal cone is

$$N(\bar{x}; \Omega) = \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} \Big[\mathrm{cone}(x - \Pi(x, \Omega)) \Big],$$

where $\Pi(x, \Omega) = \big\{ w \in \Omega \text{ s.t. } |x - w| = \mathrm{dist}(x, \Omega) \big\}$ is the Euclidean projector of x on Ω, and where the symbol “cone" stands for the conic hull of the set in question.

It is easy to check that both $\widehat{N}(\bar{x};\Omega)$ and $N(\bar{x};\Omega)$ are cones. However, the set $\widehat{N}(\bar{x};\Omega)$ is convex, while $N(\bar{x};\Omega)$ is not in general. Furthermore, $N(\bar{x};\Omega)$ cone reduces to the classical normal cone of convex analysis when Ω is convex, which it may take nonconvex values in rather simple situations as, e.g., for the graph of the function $|x|$ and the epigraph of the function $-|x|$ on $\mathbb{R}$. Nevertheless the normal cone (2.11) and the related generalized differential constructions for functions and mappings enjoy comprehensive calculus rules based on variational/extremal principles of variational analysis; see [29, 33] and the references therein.

There are a lot of calculus results were obtained by Mordukhovich in [29], one of the most meaningful properties of the limiting normal cone is that it satisfies the *intersection rule*, see[29, Corollary 3.37].

Lemma 2.2 (basic intersection rule). *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be such that $\bar{x} \in \Omega_1 \cap \Omega_2$, and let* the normal qualification condition

$$N(\bar{x},\Omega_1) \cap (-N(\bar{x},\Omega_2)) = 0 \tag{2.12}$$

be satisfied. Then we have the inclusion

$$N(\bar{x},\Omega_1 \cap \Omega_2) \subset N(\bar{x},\Omega_1) + N(\bar{x},\Omega_2) \tag{2.13}$$

Given now an extended-real-valued and lower semicontinuous function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty,\infty]$ finite at $\bar{x}$, we define its *subdifferential* at $\bar{x}$ geometrically

$$\partial\varphi(\bar{x}) := \left\{ v \in \mathbb{R}^n \middle| \, (v,-1) \in N((\bar{x},\varphi(\bar{x})); \operatorname{epi}\varphi) \right\} \tag{2.14}$$

via the normal cone (2.11) to the epigraphical set

$$\operatorname{epi}\varphi := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R}|\ \alpha \geq \varphi(x)\}$$

of φ. The reader can find in [29, 33] various analytical representations and properties of the subgradient mapping $\partial\varphi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ used in what follows.

We recall also the *symmetric subdifferential* construction for a continuous function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ defined by

$$\partial^0\varphi(\bar{x}) := \partial\varphi(\bar{x}) \cup (-\partial(-\varphi)(\bar{x})) \tag{2.15}$$

and employed in Section 3.4 for expressing necessary optimality conditions for equality constraints. Note the symmetry relation

$$\partial^0(-\varphi)(\bar{x}) = -\partial^0\varphi(\bar{x}),$$

which does not hold for the unilateral subdifferential construction (2.14).

Given further a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, define its *coderivative* [29] at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ by

$$D^*F(\bar{x},\bar{y})(v) := \Big\{u \in \mathbb{R}^n \Big|\ (u,-v) \in N\big((\bar{x},\bar{y}); \operatorname{gph} F\big)\Big\}, \quad v \in \mathbb{R}^m, \tag{2.16}$$

generated by the normal cone (2.11) to the graph $\operatorname{gph} F$. The set-valued mapping $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is set-valued mapping and clearly positive-homogeneous; Moreover, if the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is single-valued (then we drop $\bar{y} = F(\bar{x})$ in the coderivative notation) and *strictly differentiable*(smooth) around $\bar{x}$ (which is automatic when it is $\mathcal{C}^1$ around this point), then the coderivative (2.16) is also single-valued and reduces to the *adjoint* derivative operator

$$D^*F(\bar{x})(v) = \{\nabla F(\bar{x})^T v\}, \quad v \in \mathbb{R}^m.$$

It is worth noting that the coderivative values in (2.16) are often nonconvex sets due to the intrinsic nonconvexity of the normal cone on the right-hand side therein. Observe furthermore that this nonconvex normal cone is taken to a *graphical* set. Thus its convexification in (2.16), which reduces to the convexified/Clarke normal cone to the set in question, creates serious troubles; see Rockafellar [33] and Mordukhovich [29, Subsection 3.2.4] for more details.

In the general nonsmooth and/or set-valued case, the coderivative (2.16) is a positive homogeneous multifunction, which enjoys comprehensive calculus rules based on the variational and extremal principle of variational analysis; see [29, 33]. The results we need in what follows in known as the *coderivative/Mordukhovich criterion* (see [28, Theorem 5.7] and [33, Theorem 9.40] with the references therein): If F is close-graph around $(\bar{x}, \bar{y})$, then it is Lipschitz-like around this point if and only if

$$D^*F(\bar{x}, \bar{y})(0) = \{0\}. \tag{2.17}$$

Moreover if F is locally Lipschitzian, the following result is obtained

Lemma 2.3 *Let F be of closed graph and bounded around $\bar{x}$ with $F(\bar{x}) \neq \emptyset$. Then each of the following conditions is necessary and sufficient for F to be locally Lipschitzian around this point:* **(i)** *there exist a neighborhood U of $\bar{x}$ and a constant $l \geq 0$ such that*

$$\sup\left\{|x^*| : x^* \in D^*F(x,y)(y^*)\right\} \leq l|y^*| \quad \textit{for all } x \in U,\ y \in F(x),\ y^* \in \mathbb{R}^m \tag{2.18}$$

(ii) $D^*F(\bar{x},\bar{y})(0) = \{0\}$, *for all* $\bar{y} \in F(\bar{x})$.

Finally in the chapter, we recall the necessary optimality conditions of Lagrange type for the mathematical programming problems (MP) with operator and many geometric and functional constraints. Consider the following problem of mathematical programming (MP) with finitely many functional and geometric constraints. Given $\phi_j : \mathbb{R}^d \to \mathbb{R}$ for $j = 0, \ldots, s$, $g_j : \mathbb{R}^d \to \mathbb{R}^n$ for $j = 0, \ldots, p$, and $\Delta_j \subset \mathbb{R}^d$ for $j = 0, \ldots, q$, we define (MP) by

$$\begin{aligned}
&\text{minimize } \ \phi_0(z) \ \text{ subject to}\\
&\phi_j(z) \le 0 \ \text{ for } \ j = 0, \ldots, s,\\
&g_j(z) = 0 \ \text{ for } \ j = 0, \ldots, p,\\
&z \in \Delta_j \ \text{ for } \ j = 0, \ldots, q.
\end{aligned}$$

The next result gives us necessary optimality conditions for local minimizers of problem (MP) in the setting needed for the subsequent application to deriving optimality conditions in the discrete approximation problems (P_k). We express these conditions via our basic normal cone (2.11) and subdifferential (2.14) constructions above.

Lemma 2.4 (generalized Lagrange multiplier rule for mathematical programs). *Let $\bar{z}$ be a local optimal solution to problem (MP). Assume that the functions ϕ_j are Lipschitz continuous around $\bar{z}$, the mappings g_j are continuous differentiable around $\bar{z}$, and the sets Δ_j are locally closed around this point. Then there exist nonnegative numbers μ_j for $j = 0, \ldots, s$ as well as vectors $\psi_j \in \mathbb{R}^n$ for $j = 0, \ldots, p$ and $z_j^* \in \mathbb{R}^d$ for $j = 0, \ldots, q$, not equal to zero simultaneously, such that we have the conditions*

$$z_j^* \in N(\bar{z}; \Delta_j), \quad j = 0, \ldots, q,$$

$$\mu_j \phi_j(\bar{z}) = 0, \quad j = 1, \ldots, s,$$

$$-z_0^* - \ldots - z_q^* \in \partial\Big(\sum_{j=0}^{s} \mu_j \phi_j\Big)(\bar{z}) + \sum_{j=0}^{p} (\nabla g_j(\bar{z}))^T \psi_j,$$

where the symbol "A^T" indicates the matrix transposition.

Proof. This result follows from necessary optimality conditions given [30, Theorem 5.21] for problems with a single geometric constraint and the basic intersection rule for the normal cone (2.11) taken from [29, Theorem 3.4]. □

Chapter 3

Implicit Euler Approximation of One-sided Lipschitzian Differential Inclusions

3.1 Introduction

This chapter concerns the study of the following generalized Bolza problem $(\widetilde{P})$ governed by differential inclusions with the geometric and functional endpoint constraints:

$$\text{minimize } J[x] := \varphi_0\big(x(T)\big) + \int_0^T f\big(x(t), \dot{x}(t), t\big)dt \tag{3.1}$$

over absolutely continuous trajectories $x : [0,T] \to \mathbb{R}^n$ satisfying the differential inclusion

$$\dot{x}(t) \in F\big(x(t), t\big) \ \text{ a.e. } \ t \in [0,T] \ \text{ with } \ x(0) = x_0 \in \mathbb{R}^n \tag{3.2}$$

subject to the geometric and functional endpoint constraints given by, respectively,

$$x(T) \in \Omega \subset \mathbb{R}^n, \tag{3.3}$$

$$\varphi_i(x(T)) \le 0 \ \text{ for } \ i = 1, \dots, m, \tag{3.4}$$

$$\varphi_i(x(T)) = 0 \ \text{ for } \ i = m+1, \dots, m+r. \tag{3.5}$$

Here x_0 is a fixed n-vector, $F\colon \mathbb{R}^n \times [0,T] \rightrightarrows \mathbb{R}^n$ is a set-valued mapping/multifunction, Ω is an nonempty set, f and φ_i for $i = 0, \dots, m+r$ are real-valued functions.

In this chapter we exploit a weaker property than MOSL known as the *relaxed one-sided Lip-*

schitz (ROSL) condition; see below. The ROSL property of set-valued mappings was introduced by Tzanko Donchev in [8] under a different name and has already been employed in the studies of various aspects of analysis of set-valued mappings, differential inclusions, and their discrete approximations; see, e.g., [4, 9, 10, 12]. In particular, the paper [10] contains an extension to the ROSL case of the fundamental Filippov theorem about relationships between trajectories and "quasitrajectories" of Lipschitzian differential inclusions and provides applications of this result to the stability analysis of the explicit Euler scheme. In [4], similar and related solvability and stability results were developed for the parameterized *implicit Euler scheme*

$$\Phi_h(x) := \{y \in \mathbb{R}^n |\ y \in x + hF(y)\}, \quad h > 0, \tag{3.6}$$

generated by ROSL mappings F with compact and convex values. Note that the implicit framework of (3.6) is essentially more involved in comparison with the explicit one

$$\Phi_h(x) := \{y \in \mathbb{R}^n |\ y \in x + hF(x)\}, \quad h > 0, \tag{3.7}$$

studied and applied in [9, 10, 12] and other publications.

The main goal of this paper is to use the implicit Euler scheme (3.6) to construct and investigate the *discrete approximations*

$$x^k_{j+1} \in x^k_j + h_k F(x^k_{j+1}, t_{j+1}), \quad k \in I\!N := \{1, 2, \ldots\} \text{ with } h_k \downarrow 0 \text{ as } k \to \infty, \tag{3.8}$$

of the ROSL differential inclusion (3.2) and the generalized Bolza problem $(\widetilde{P})$ for it with establishing the strong convergence of discrete approximations (in the sense specified below) and deriving necessary optimality conditions for their optimal solutions. To the best of our knowledge, the results obtained in what follows are new for discrete approximations constructed via the implicit Euler scheme even for the case of unconstrained differential inclusions satisfying the classical Lipschitz condition with respect to state variables.

We develop new results in the aforementioned directions outlined in what follows. Section 3.2 presents a constructive procedure allowing us to *strongly approximate* in the norm topology of $C[0,T]$ a given feasible trajectory $\bar{x}(\cdot)$ of the differential inclusion (3.2) by feasible solutions to the *implicit Euler* finite-difference inclusions (3.8) piecewise linearly extended to $[0,T]$. Furthermore, we justify here even *stronger* $W^{1,2}[0,T]$-norm approximation of $\bar{x}(\cdot)$ by feasible extended discrete trajectories in the following two major cases: either F is *ROSL and locally graph-convex*, or F is *locally Lipschitzian*. Some counterparts of this result involving the *explicit* Euler scheme (3.7) can be found (with different proofs) in [27, 30] for Lipschitzian differential inclusions and in [11] for those satisfying the MOSL condition. We are not familiar with any results of this type (involving either the $C[0,T]$ or $W^{1,2}[0,T]$ convergence) for discrete approximations of differential inclusions based on the implicit Euler scheme.

In Section 3.3 we construct a sequence of finite-difference Bolza type problems $(\widetilde{P}_k)$ as $k \in I\!N$ with the dynamic constraints given by the implicit Euler scheme (3.8) under appropriate approximations of the cost functional (3.1) and the endpoint constraints in (3.3)–(3.5). Then we show that optimal solutions to $(\widetilde{P}_k)$ and their slight modifications exist for all large $k \in I\!N$ and *norm converge* in the $C[0,T]$ topology for the case of *strong local minimizers* of $(\widetilde{P})$ in the general ROSL setting and the $W^{1,2}[0,T]$ topology in the case of *intermediate local minimizers* of $(\widetilde{P})$ under the additional assumptions on these minimizers imposed in Section 3.2. The obtained results seem to be the first achievements in this direction for the *implicit* Euler scheme (3.8). It is worth mentioning however that our approach to the strong approximation and convergence results obtained in Sections 3.2 and 3.3 require, along with the ROSL condition on the differential inclusion, the *unform boundedness* of the velocity sets. This does not allow us to cover the corresponding developments presented of [6, 7] for discrete approximations of control problems governed by Moreau's sweeping process, which is described by a dissipative while intrinsically unbounded differential inclusion studied in [6] via the explicit Euler scheme by exploiting certain specific features of the sweeping process generated by controlled moving sets.

In addition to the well-posedness results for discrete approximations of $(\widetilde{P})$ via the implicit Euler scheme obtained in Sections 3.2 and 3.3, we derive in Section 3.4 under fairly mild assumptions necessary conditions for optimal solutions to the nonsmooth discrete approximations problems $(\widetilde{P}_k)$ associated with the implicit discrete inclusions (3.8) that are different from necessary optimality conditions for the corresponding problems associated with explicit Euler counterparts. Due to the established convergence of discrete optimal solutions, the necessary optimality conditions for problems $(\widetilde{P}_k)$ obtained in this way can be treated as *suboptimality* (almost optimality) conditions for the original Bolza problem $(\widetilde{P})$ and can be also viewed as a certain justification of *numerical algorithms* based on discrete approximations. The final Section 3.5 presents necessary optimality conditions for relaxed intermediate minimizers to the original Bolza problem $(\widetilde{P})$ by passing to the limit from those obtained necessary conditions for discrete problems in Section 3.4.

Finally, section 3.5 is devoted to the limiting procedure in discrete approximations that allows us to derive necessary optimality conditions for an i.r.l.m. to the original Bolza problem $(\widetilde{P})$. The obtained results on the well-posedness of discrete approximations and necessary optimality conditions allow us to justify a numerical approach to solve the generalized Bolza problem for one-sided Lipschitzian differential inclusions by using discrete approximations constructed via the implicit Euler scheme.

3.2 Strong Approximation via Implicit Euler Scheme

In this section we justify the possibility to *strongly approximate* (in the norm topology of either $C[0,T]$ or $W^{1,2}[0,T]$) feasible trajectories of the ROSL inclusion (3.2) constructed via the implicit Euler scheme. Given an *arbitrary* trajectory $\bar{x}(\cdot)$ of (3.2), we impose the following assumptions of F near $\bar{x}(\cdot)$ standing throughout Sections 3.2 and 3.3. For simplicity, suppose that the uniform boundedness and ROSL moduli below are constant on $[0,T]$. They can obviously be replaced by the continuous functions $m_F(t)$ and $l(t)$ on this compact interval while the proofs of the main results presented in Sections 3.2 and 3.3 can be modified to more general cases of the Riemann and Lebesgue integrability.

(H1) There exists an open set $U \subset \mathbb{R}^n$ and a number $m_F > 0$ such that $\bar{x}(t) \in U$ for all $t \in [0,T]$ and the multifunction $F\colon U \times [0,T] \to \mathcal{CC}(\mathbb{R}^n)$ from (3.2) satisfies the *uniform boundedness condition*

$$F(x,t) \subset m_F \mathbb{B} \;\text{ for all }\; x \in U, \quad \text{a.e. } \; t \in [0,T].$$

(H2) Given U from (H1), for all $x_1, x_2 \in U$, a.e. $t \in [0,T]$, and $y_1 \in F(x_1,t)$ there exists $y_2 \in F(x_2,t)$ such that we have the *relaxed one-sided Lipschitzian condition*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \le l|x_1 - x_2|^2.$$

(H3) The multifunction $F(\cdot,t)$ is *continuous* on the neighborhood U from (H1) for a.e. $t \in [0,T]$

while $F(x,\cdot)$ is *a.e. continuous* on $[0,T]$ uniformly in $x \in U$ with respect to the Pompieu-Hausdorff metric.

We now construct a finite-difference approximation of the differential inclusion in (3.2) by using the *implicit Euler method* to replace the time derivative by

$$x(t+h) \in x(t) + hF(x(t+h),t) \ \text{ as } \ h \downarrow 0.$$

To formalize this process, for any $k \in I\!N$ define the discrete grid/mesh on $[0,T]$ by $T_k := \big(t_j \big|\, j = 0,1,\ldots,k\big)$ with $t_0 := 0$, $t_k := T$, and stepsize

$$h_k := T/k = t_{j+1} - t_j.$$

Then the corresponding discrete inclusions associated with (3.2) via the implicit Euler scheme are constructed as follows:

$$x^k_{j+1} \in x^k_j + h_k F(x^k_{j+1}, t_{j+1}) \ \text{ for } \ j = 0,\ldots,k-1, \tag{3.9}$$

where the starting vector x_0 in (3.9) is taken from (3.2).

The next theorem justifies the aforementioned strong $W^{1,2}[0,T]$-approximation of feasible solutions to (3.2) by those for the discrete inclusions (3.9).

Theorem 3.1 (discrete approximation of ROSL differential inclusions). *Let $\bar{x}(\cdot)$ be a feasible trajectory for* (3.2) *such that $\dot{\bar{x}}(t)$ is Riemann integrable on $[0,T]$ and the standing assumptions* (H1)–(H3) *are satisfied. Then the following assertions hold:*

(i) *There is a sequence $\{z^k_j | j = 0,\ldots,k\}$ of feasible solutions to the discrete inclusions* (3.9) *such that their piecewise linearly extensions to* $[0,T]$ *converge to* $\bar{x}(t)$ *uniformly on* $[0,T]$, *i.e., in the norm topology of* $C[0,T]$.

(ii) *Assume in addition that either the graph of* $F(\cdot,t)$ *is locally convex around* $(\bar{x}(t),\dot{\bar{x}}(t))$, *or* $F(\cdot,t)$ *is locally Lipschitzian around* $\bar{x}(t)$ *for a.e.* $t\in[0,T]$. *Then there is a sequence* $\{z_j^k | j=0,\ldots,k\}$ *of feasible solutions to* (3.9) *such that the piecewise constantly extended to* $[0,T]$ *discrete velocity functions*

$$v^k(t) := \frac{z_{j+1}^k - z_j^k}{h_k}, \quad t\in(t_j,t_{j+1}], \quad j=0,\ldots,k-1, \tag{3.10}$$

converge to $\dot{\bar{x}}(\cdot)$ *as* $k\to\infty$ *in the norm topology of* $L^2[0,T]$, *which is equivalent to the* $W^{1,2}[0,T]$*-norm convergence of the piecewise linear functions* $z^k(t)$ *represented by*

$$z^k(t) = x_0 + \int_0^t v^k(s)\,ds \;\; \textit{for all} \;\; t\in[0,T], \quad k=1,2,\ldots. \tag{3.11}$$

Proof. Fix an arbitrary feasible trajectory $\bar{x}(t)$ for (3.2) from the formulation of the theorem and denote $\bar{x}_j := \bar{x}(t_j)$. Taking into account the density of step functions in $L^1[0,T]$, we can find without loss of generality a sequence of functions $w^k(\cdot)$ on $[0,T]$ such that $w^k(t)$ are constant on $(t_j,t_{j+1}]$ and $w^k(t)$ converge to $\dot{\bar{x}}(t)$ as $k\to\infty$ in the norm topology of $L^1[0,T]$. It follows from (H1) that

$$|w^k(t)| \le m_F + 1 \;\text{ for all }\; t\in[0,T] \;\text{ and }\; k\in I\!N.$$

Define further the sequences

$$w_j^k := w^k(t_j) \text{ for } j = 1, \ldots, k \text{ and } \xi_k := \int_0^T |\dot{\bar{x}}(t) - w^k(t)|\, dt \to 0 \text{ as } k \to \infty \tag{3.12}$$

and for each $k \in I\!N$ form recurrently the collection of vectors $\{y_0^k, \ldots, y_k^k\}$ by

$$y_{j+1}^k := y_j^k + h_k w_{j+1}^k \text{ for } j = 0, \ldots, k-1 \text{ with } y_0^k = x_0. \tag{3.13}$$

Note that the continuous-time vector functions

$$y^k(t) := x_0 + \int_0^t w^k(s)\, ds, \quad 0 \le t \le T,$$

are piecewise linear extensions of the discrete ones (3.13) on $[0, T]$ satisfying the estimate

$$|y^k(t) - \bar{x}(t)| \le \int_0^t |w^k(s) - \dot{\bar{x}}(s)|\, ds \le \xi_k \text{ for all } t \in [0, T],\ k \in I\!N, \tag{3.14}$$

where ξ_k is taken from (3.12). Now we construct a sequence of discrete trajectories for (3.9) by the following *algorithmic procedure*.

To define such trajectories $z^k = (z_0^k, \ldots, z_k^k)$ of (3.9), put $z_0^k := x_0$ and suppose that the vectors z_j^k have been already found. Then for any $k \in I\!N$ sufficiently large (i.e., when h_k is small) we use the solvability result from Lemma 2.1 valid under assumptions (H2) and (H3) and solve the discrete inclusions (3.9) for z_{j+1}^k. Taking into account the error estimate in Lemma 2.1, the construction of y_j^k in (3.13), and the corresponding properties of the distance (2.1), we deduce that the vector z_{j+1}^k satisfies the discrete inclusion

$$z_{j+1}^k \in z_j^k + h_k F(z_{j+1}^k, t_{j+1}) \tag{3.15}$$

and the following relationships for each $j \in \{1, \dots, k-1\}$ and all (large) $k \in I\!N$:

$$\begin{aligned}
|z_{j+1}^k - y_{j+1}^k| &\leq \frac{1}{1-lh_k}\text{dist}\big(y_{j+1}^k, z_j^k + h_k F(y_{j+1}^k, t_{j+1})\big) \\
&\leq \frac{1}{1-lh_k}\text{dist}(y_{j+1}^k, y_j^k + h_k F(y_{j+1}^k, t_{j+1})) \\
&+ \frac{1}{1-lh_k}\text{dist}\big(y_j^k + h_k F(y_{j+1}^k, t_{j+1}), z_j^k + h_k F(y_{j+1}^k, t_{j+1})\big) \\
&\leq \frac{|z_j^k - y_j^k|}{1-lh_k} + \frac{h_k}{1-lh_k}\text{dist}\Big(\frac{y_{j+1}^k - y_j^k}{h_k}, F(y_{j+1}^k, t_{j+1})\Big) \\
&= \frac{|z_j^k - y_j^k|}{1-lh_k} + \frac{h_k}{1-lh_k}\text{dist}\big(w_{j+1}^k, F(y_{j+1}^k, t_{j+1})\big).
\end{aligned}$$

Thus we arrive at the estimate valid for all $j = 0, \dots, k-1$ and $k \in I\!N$:

$$|z_{j+1}^k - y_{j+1}^k| \leq \frac{|z_j^k - y_j^k|}{1-lh_k} + \frac{h_k}{1-lh_k}\text{dist}\big(w_{j+1}^k, F(y_{j+1}^k, t_{j+1})\big). \tag{3.16}$$

Proceeding further by induction implies that

$$|z_{j+1}^k - y_{j+1}^k| \leq h_k \sum_{m=1}^{j+1} \Big(\frac{1}{1-lh_k}\Big)^{j+2-m}\text{dist}\big(w_m^k, F(y_m^k, t_m)\big),$$

which yields by choosing $k \in I\!N$ with $lh_k < 1/2$ that

$$\begin{aligned}
|z_{j+1}^k - y_{j+1}^k| &\leq h_k \sum_{m=1}^{j+1} (1+2lh_k)^{j+2-m}\text{dist}\big(w_m^k, F(y_m^k, t_m)\big) \\
&\leq h_k e^{2lT} \sum_{m=1}^{j+1} \text{dist}(w_m^k, F(y_m^k, t_m)).
\end{aligned} \tag{3.17}$$

We recall next the average modulus of continuity of F defined by

$$\tau(F;h) := \sup{}_{x\in U} \int_0^T \sup\left\{\mathrm{dist}(F(x,t'),F(x,t''))\Big|\, t',t'' \in \left[t-\frac{h}{2}, t+\frac{h}{2}\right]\right\} dt$$

and consider the quantities ζ_k with the estimates

$$\begin{aligned}\zeta_k \quad &:= \sum_{m=1}^{k} h_k \mathrm{dist}\big(w_m^k, F(y_m^k,t_m)\big) = \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m} \mathrm{dist}(w_m^k, F(y_m^k,t_m)) \\ &\le \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m} \mathrm{dist}\big(w_m^k, F(y_m^k,t)\big) + \tau(F;h_k), \quad k \in I\!N.\end{aligned} \tag{3.18}$$

It is well known (see, e.g., [30, Proposition 6.3]) that the a.e. continuity of $F(x,\cdot)$ on $[0,T]$ uniformly in $x \in U$ assumed in (H3) is equivalent to the convergence $\tau(F;h_k) \to 0$.

Let us show next that $\displaystyle\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m} \mathrm{dist}(w_m^k, F(y_m^k,t)) \to 0$ as $k \to \infty$. Taking into account that $\bar{x}(\cdot)$ is a feasible trajectory for the differential inclusion (3.2) and that $w^k(\cdot) \to \dot{\bar{x}}(\cdot)$ strongly in $L^1[0,T]$ and remembering also that for each $k \in I\!N$ the functions $w^k(t)$ are constant on the intervals $(t_{m-1}, t_m]$, $m = 1,2,\ldots$, and that $\dot{\bar{x}}(t)$ is Riemann integrable (that is, a.e. continuous) on $[0,T]$, we can find $\widetilde{t}_m \in (t_{m-1}, t_m]$ such that

$$\dot{\bar{x}}(\widetilde{t}_m) \in F(\bar{x}(\widetilde{t}_m), \widetilde{t}_m) \;\text{ and }\; \sum_{m=1}^{k}\int_0^T |\dot{\bar{x}}(\widetilde{t}_m) - w^k(t)|\,dt \le 2\xi_k.$$

This readily leads us to the following inequalities:

$$\begin{aligned}
&\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\operatorname{dist}(w_m^k,F(y_m^k,t))dt\\
&\le \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[\operatorname{dist}(w_m^k,F(\bar{x}_m,t))+\operatorname{dist}(F(\bar{x}_m,t),F(y_m^k,t))\Big]dt\\
&\le \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[\operatorname{dist}(w_m^k,F(\bar{x}(\widetilde{t}_m),\widetilde{t}_m))+\operatorname{dist}(F(\bar{x}_m,t),F(\bar{x}(\widetilde{t}_m),\widetilde{t}_m))\\
&\qquad\qquad +\operatorname{dist}(F(\bar{x}_m,t),F(y_m^k,t))\Big]dt\\
&\le \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[|w^k(t)-\dot{\bar{x}}(\widetilde{t}_m)|+\operatorname{dist}(F(\bar{x}_m,t),F(\bar{x}(\widetilde{t}_m),\widetilde{t}_m))\\
&\qquad\qquad +\operatorname{dist}(F(\bar{x}_m,t),F(y_m^k,t))\Big]dt\\
&\le 2\xi_k+\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[\operatorname{dist}(F(\bar{x}_m,t),F(\bar{x}(\widetilde{t}_m),\widetilde{t}_m))+\operatorname{dist}(F(\bar{x}_m,t),F(y_m^k,t))\Big]dt\\
&\le 2\xi_k+\tau(F;h_k)+\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[\operatorname{dist}(F(\bar{x}(t_m),t_m),F(\bar{x}(\widetilde{t}_m),\widetilde{t}_m))\\
&\qquad\qquad\qquad +\operatorname{dist}(F(\bar{x}_m,t),F(y_m^k,t))\Big]dt.
\end{aligned}$$

Under the assumption (H3) we have $\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\operatorname{dist}(w_m^k,F(y_m^k,t))\to 0$. By employing (3.12) and the definition of ζ_k in (3.18), it gives us the convergence $\zeta_k\to 0$ as $k\to\infty$.

Using this and the last inequality in (3.17) allows us to conclude that

$$|z_{j+1}^k-y_{j+1}^k|\le \zeta_k e^{2lT}\ \text{ for all }\ j=0,\dots,k-1\ \text{ and all }\ k\in I\!N. \tag{3.19}$$

Furthermore, we easily get the estimates

$$|z_{j+1}^k-\bar{x}_{j+1}|\le \zeta_k e^{2lT}+|y_{j+1}^k-\bar{x}_{j+1}|\le \zeta_k e^{2lT}+\xi_k=:\eta_k, \tag{3.20}$$

where $\eta_k\to 0$ due to (3.12) and $\zeta_k\to 0$ as $k\to\infty$.

Considering next the the piecewise linear functions $z^k(\cdot)$ built in (3.11) by using the discrete velocity $v^k(\cdot)$ from (3.10), we get from (3.10) and (3.15) that

$$\dot{z}^k(t) = v_j^k = \frac{z_j^k - z_{j-1}^k}{h_k} \in F\big(z^k(t_j), t_j\big) \ \text{ on } \ (t_{j-1}, t_j], \quad j = 1, \ldots, k.$$

It follows from the uniform boundedness of F in (H1) that there is a subsequence of $\{\dot{z}^k(\cdot)\}$ (without relabeling) that converges to some function in $L^1[0,T]$, which cannot be anything but $\dot{\bar{x}}(t)$ due to the relationships in (3.20) established above. Thus $\dot{z}^k(\cdot) \to \dot{\bar{x}}(\cdot)$ *weakly* in $L^1[0,T]$ as $k \to \infty$. The latter is equivalent to the *uniform convergence* $z^k(\cdot) \to \bar{x}(t)$ by the Newton-Leibniz formula (3.11), and so we get (i).

Now we justify assertion (ii) proving that in fact $\dot{z}^k(\cdot) \to \dot{\bar{x}}(\cdot)$ as $k \to \infty$ *strongly* in $L^1[0,T]$ provided that either the graph of $F(\cdot,t)$ is convex around $(\bar{x}(t), \dot{\bar{x}}(t))$, or the mapping $x \mapsto F(x,t)$ is locally Lipschitzian around $\bar{x}(t)$ for a.e. $t \in [0,T]$.

First we examine the case when the *graph* of $F(\cdot,t)$ is *locally convex*. The classical Mazur's weak closure theorem tells us that there is a sequence of *convex combinations* of $\dot{z}^k(\cdot)$, which converges to $\dot{\bar{x}}(\cdot)$ in the norm topology of $L^1[0,T]$ and thus contains a subsequence (no relabeling) converging to $\dot{\bar{x}}(t)$ for a.e. $t \in [0,T]$. Taking into account the graph convexity of $F(\cdot,t)$ and the piecewise constant nature of $\dot{z}^k(t)$ and assuming without loss of generality that for each element of the sequence of convex combinations the corresponding partition of the interval is a subpartition of the previous one, we conclude that all the elements of the aforementioned sequence of convex combinations are *feasible* trajectories of the discrete approximation systems for any $k \in I\!N$. Therefore we get a sequence of feasible solutions to the discrete inclusions (3.9) whose piecewise linear extensions on $[0,T]$ converges to $\dot{\bar{x}}(\cdot)$ strongly in $L^1[0,T]$. Keeping for simplicity the notation $\dot{z}^k(\cdot)$ for the elements of this sequence allows us to write

$$\alpha_k := \int_0^T |\dot{z}^k(t) - \dot{\bar{x}}(t)|\,dt \to 0 \ \text{ as } \ k \to \infty. \tag{3.21}$$

To complete the proof of the theorem in the convex graph case, it remains to verify the the convergence of $\{\dot{z}^k(\cdot)\}$ to $\dot{\bar{x}}(\cdot)$ in the norm topology of $L^2[0,T]$. By the constructions above and assumption (H1), it is implied by the following relationships:

$$\begin{aligned}
&\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big|\frac{z^k(t_j) - z^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \\
&\qquad = \sum_{j=1}^{k} \max\Big(|v_j^k| + |\dot{\bar{x}}(t)|\Big) \int_{t_{j-1}}^{t_j} |v_j^k - \dot{\bar{x}}(t)|\,dt \\
&\qquad \le 2m_F \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |v_j^k - \dot{\bar{x}}(t)|\,dt = 2m_F\alpha_k,
\end{aligned} \tag{3.22}$$

where α_k is taken from (3.21). This justifies the $W^{1,2}[0,T]$-norm convergence of the extended discrete trajectories $\{z^k(\cdot)\}$ from (3.11) in the first case under consideration.

Let us finally consider the other case in (ii) when $F(\cdot,t)$ is *locally Lipschitzian* around $\bar{x}(t)$ for a.e. $t \in [0,T]$. Then for all $j \in \{1,\ldots,k-1\}$ we have the estimates

$$\begin{aligned}
|z_{j+1}^k - z_j^k - h_k w_{j+1}| &\le \frac{1}{1-lh_k}\text{dist}(z_j^k + h_k w_{j+1}, z_j^k + h_k F(z_j^k + h_k w_{j+1}, t_{j+1})) \\
&\le \frac{1}{1-lh_k}\text{dist}(h_k w_{j+1}, h_k F(z_j^k + h_k w_{j+1}, t_{j+1})) \\
&\le \frac{h_k}{1-lh_k}\Big[\text{dist}(w_{j+1}, F(y_j^k + h_k w_{j+1}, t_{j+1})) \\
&+\text{dist}(F(y_j^k + h_k w_{j+1}, t_{j+1}), F(z_j^k + h_k w_{j+1}, t_{j+1}))\Big] \\
&\le \frac{h_k}{1-lh_k}\Big[l|z_j^k - y_j^k| + \text{dist}(w_{j+1}, F(y_{j+1}, t_{j+1}))\Big].
\end{aligned} \tag{3.23}$$

Combining (3.23) with (3.19) gives us

$$\begin{aligned}\int_0^T |\dot{z}^k(t) - \dot{y}^k(t)|\,dt &= \sum_{j=1}^{k} h_k |v_j^k - w_j^k| \\ &\le \frac{1}{1-lh_k}\Big[lh_k \sum_{j=1}^{k} |z_{j-1}^k - y_{j-1}^k| + \sum_{j=1}^{k} h_k \text{dist}(w_j^k, F(y_j^k, t_j))\Big] \\ &\le 2(lT\zeta_k e^{2lT} + \zeta_k) = 2\zeta_k(lTe^{2lT}+1) \to 0 \quad \text{as} \quad k\to\infty,\end{aligned}$$

where $\zeta_k \to 0$ is taken from (3.18). Taking further into account that

$$\int_0^T |\dot{z}^k(t) - \dot{\bar{x}}(t)|\,dt \le \int_0^T |\dot{z}^k(t) - \dot{y}^k(t)|\,dt + \int_0^T |\dot{y}^k(t) - \dot{\bar{x}}(t)|\,dt$$

and using the convergence $\xi_k \to 0$ in (3.12) with $\dot{y}^k(t) = w^k(t)$ tell us that the $L^1[0,1]$-norm convergence of $\{\dot{z}^k(\cdot)\}$ in (3.21) holds in the second case under consideration. Applying now (3.22) justifies (ii) in this case and thus completes proof of the theorem. □

3.3 Strong Convergence of Discrete Optimal Solutions

In this section we construct a sequence of well-posed discrete approximations of the Bolza problem $(\widetilde{P})$ for ROSL differential inclusions and justify the norm convergence in either $C[0,T]$ or $W^{1,2}[0,T]$ topology of their optimal solutions to either a strong local minimizer or an intermediate relaxed local minimizer $\bar{x}(\cdot)$ of $(\widetilde{P})$, respectively. In addition to our standing assumptions (H1)–(H3) on the right-hand side F in (3.2) and those (if needed) from Theorem 3.1 formulated now around the given local minimizer, the following ones are imposed here on the functions f and φ_0 in the Bolza cost functional (3.1) as well as on the functions φ_i, $i = 1,\ldots,m+r$, and the set Ω in the endpoint constraints (3.3)–(3.5).

(H4) The function $f(x,v,\cdot)$ is a.e. continuous on $[0,T]$ and bounded uniformly in $(x,v) \in$

$U \times (m_F B)$. Furthermore, there exists $\nu > 0$ such that the function $f(\cdot,\cdot,t)$ is continuous on the set

$$A_\nu(t) = \big\{(x,v) \in U \times (m_F + \nu)\mathbb{B} \big| \; v \in F(x,t') \;\text{ for some }\; t' \in (t-\nu, t]\big\}$$

uniformly in t on the interval $[0,T]$.

(H5) The cost function φ_0 is continuous on U, while the constraint functions φ_i are Lipschitz continuous on U for all $i = 1,\ldots,m+r$. Furthermore, the endpoint constraint set Ω is locally closed around $\bar{x}(T)$.

Given a *r.i.l.m.* $\bar{x}(\cdot)$ in (P), suppose without loss of generality (due to (H1)) that $\alpha = 1$ and $p = 2$ in (2.5) and the definition of r.i.l.m.. Denote by $L > 0$ a common Lipschitz constant for the functions φ_i, $i = 1,\ldots,m+r$, on U and take the sequence $\{\eta_k\}$ in (3.20) constructed via the approximation of the local optimal solution $\bar{x}(\cdot)$ under consideration. Then we define a sequence of discrete approximation problems $(\widetilde{P}_k)$, $k \in \mathbb{N}$, as follows:

$$\begin{aligned} \text{minimize } J_k[x^k]: \; &= \varphi_0(x^k(t_k)) + h_k \sum_{j=1}^{k} f\Big(x^k(t_j), \frac{x^k(t_j) - x^k(t_{j-1})}{h_k}, t_j\Big) \\ &+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big|\frac{x^k(t_j) - x^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \end{aligned} \tag{3.24}$$

over trajectories $x^k = (x_0^k,\ldots,x_k^k)$ of the discrete inclusions (3.9) subject to the constraints

$$|x^k(t_j) - \bar{x}(t_j)|^2 \le \frac{\varepsilon^2}{4} \;\text{ for }\; j = 1,\ldots,k, \tag{3.25}$$

$$\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big|\frac{x^k(t_j) - x^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \le \frac{\varepsilon}{2}, \tag{3.26}$$

$$x_k^k \in \Omega_k := \Omega + \eta_k I\!B, \tag{3.27}$$

$$\varphi_i(x_k^k) \le L\eta_k \ \text{ for } \ i = 1, \dots, m, \tag{3.28}$$

$$-L\eta_k \le \varphi_i(x_k^k) \le L\eta_k \ \text{ for } \ i = m+1, \dots, m+r, \tag{3.29}$$

where $\varepsilon > 0$ is fixed and taken from (2.5) for the given r.i.l.m. $\bar{x}(\cdot)$.

If $\bar{x}(\cdot)$ is a given *strong local minimizer* for $(\widetilde{P})$ with $f = f(x,t)$, we construct a simplified sequence of discrete approximations problems $(\check{P}_k)$ as follows:

$$\text{minimize } \check{J}_k[x^k] := \varphi_0(x^k(t_k)) + h_k \sum_{j=1}^{k} f(x^k(t_j), t_j) + \sum_{j=1}^{k} |x^k(t_j) - \bar{x}(t_j)|^2 \tag{3.30}$$

subject to the constraints (3.25)–(3.29) with η_k taken from (3.20).

The next theorem shows that problems $(\widetilde{P}_k)$ and $(\check{P}_k)$ admit optimal solutions for all large $k \in I\!N$ and that extended discrete optimal solutions to these problems converge to $\bar{x}(\cdot)$ in the corresponding norm topology of either $C[0,T]$ or $W^{1,2}[0,T]$ depending on the type of local minima (strong or intermediate) which $(\widetilde{P})$ achieves at $\bar{x}(\cdot)$.

Theorem 3.2 (strong convergence of discrete optimal solutions). *Let $\bar{x}(\cdot)$ be a Riemann integrable local optimal solution to the original Bolza problem $(\widetilde{P})$ under the validity of assumptions* (H1)-(H5) *around $\bar{x}(\cdot)$. The following assertions hold:*

(i) *If $\bar{x}(\cdot)$ is a strong local minimizer for $(\widetilde{P})$ with $f = f(x,t)$, then each problem $(\check{P}_k)$ admits an optimal solution $\bar{x}^k(\cdot)$ for large $k \in I\!N$ and the sequence $\{\bar{x}^k(\cdot)\}$ piecewise linearly extended to $[0,T]$ converges to $\bar{x}(\cdot)$ as $k \to \infty$ in the norm topology of $C[0,T]$.*

(ii) *If $\bar{x}(\cdot)$ is a r.i.l.m. in $(\widetilde{P})$ and the assumptions of Theorem* 3.1(ii) *are satisfied for $\bar{x}(\cdot)$, then each problem $(\widetilde{P}_k)$ admits an optimal solution $\bar{x}^k(\cdot)$ whenever $k \in I\!N$ is sufficiently* *large and the sequence* $\{\bar{x}^k(\cdot)\}$ *piecewise linearly extended to* $[0,T]$ *converges to* $\bar{x}(\cdot)$ *as* $k \to \infty$ *in the norm topology of* $W^{1,2}[0,T]$.

Proof. We verify the existence of optimal solutions to problems $(\widetilde{P}_k)$ and $(\check{P}_k)$ in a parallel way. Observe first that both $(\widetilde{P}_k)$ and $(\check{P}_k)$ admit feasible solutions for all $k \in I\!N$ sufficiently large. Indeed, take for each k the discrete trajectories $z^k := (z_0^k, \dots, z_k^k)$ constructed in Theorem 3.1(i) to approximate the r.i.l.m. $\bar{x}(\cdot)$ and in Theorem 3.1(ii) to approximate the r.i.l.m. $\bar{x}(\cdot)$. Then both these functions satisfy the discrete inclusion (3.9), and it remains to verify that the corresponding z^k fulfills the constraints in (3.25), (3.27)–(3.29) in the case of $(\check{P}_k)$ and those in (3.25)–(3.29) in the case of $(\widetilde{P}_k)$. The validity of (3.25) and (3.27) in both cases follows from (3.20) for large k, while the validity of the additional constraint (3.26) for $(\widetilde{P}_k)$ follows from (3.21). The fulfillment of the inequality constraints in (3.28) and (3.29) for z_k^k follows by these arguments from the validity of (3.4) and (3.5) for $\bar{x}(T)$, respectively, and the local Lipschitz continuity of the endpoint functions

$$|\varphi_i(z_k^k) - \varphi_i(\bar{x}(T))| \le L|z_k^k - \bar{x}(T)| \le L\eta_k, \quad i = 1, \dots, m + r.$$

Thus for each $k \in I\!N$ (omitting the expression "for all large k" in what follows) the sets of feasible solutions to $(\widetilde{P}_k)$ and $(\check{P}_k)$ are nonempty. It is clear from the construction of $(\widetilde{P}_k)$ and $(\check{P}_k)$ and the assumptions made that each of these sets is closed and bounded. This ensures the existence of optimal solutions to $(\widetilde{P}_k)$ by the classical Weierstrass existence theorem due to the continuity of the functions φ_0 and f in (3.24) and (3.30) .

Next we proceed with the proof of the strong $W^{1,2}[0,T]$-convergence in (ii) for any sequence of the discrete optimal solutions $\{\bar{x}^k(\cdot))\}$ in $(\widetilde{P}_k)$ piecewise linearly extended to the continuous-time interval $[0,T]$. To this end let us first show that

$$\liminf_{k\to\infty} J_k[\bar{x}^k] \le J[\bar{x}] \tag{3.31}$$

for the optimal values of the cost functional in (3.24). It follows from the optimality of $\bar{x}^k(\cdot)$ for $(\widetilde{P}_k)$ and the feasibility of $z^k(\cdot)$ taken from the proof of (i) for this problem that $J_k[\bar{x}^k] \le J_k[z^k]$ for each k. To get (3.31), it suffices to show therefore that

$$J_k[z^k] \to J[\bar{x}] \quad \text{as} \quad k \to \infty \tag{3.32}$$

including the verification of the existence of the limit. We have from (3.24) that

$$\begin{aligned} J_k[z^k] \; &= \varphi_0(z^k(t_k)) + h_k \sum_{j=1}^{k} f\Big(z^k(t_j), \frac{z^k(t_j) - z^k(t_{j-1})}{h_k}, t_j\Big) \\ &+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big|\frac{z^k(t_j) - z^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \end{aligned}$$

and deduce from the convergence $z^k(t_k) \to \bar{x}(T)$ and the continuity assumption on φ_0 in (H5) the convergence $\varphi_0(z^k(t_k)) \to \varphi_0(\bar{x}(T))$ as $k \to \infty$ of the terminal cost function in (3.24). Furthermore, it follows from (3.21) that

$$\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big|\frac{z^k(t_j) - z^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \to 0 \text{ as } \; k \to \infty.$$

To justify (3.32), we only need to check that

$$h_k \sum_{j=1}^{k} f\Big(z^k(t_j), \frac{z^k(t_j) - z^k(t_{j-1})}{h_k}, t_j\Big) \to \int_0^T f(\bar{x}(t), \dot{\bar{x}}(t), t)dt \;\; \text{as} \;\; k \to \infty.$$

The continuity assumptions on f in (H4) imply without loss of generality that

$$\Big|f(z^k(t_j),\frac{z^k(t_j)-z^k(t_{j-1})}{h_k},t_j)-f(z^k(t_j),\frac{z^k(t_j)-z_k(t_{j-1})}{h_k},t)\Big|\le\frac{\varepsilon}{T}$$

for all $k\in I\!N$ and a.e. $t\in[0,T]$. Employing now Lebesgue's dominated convergence theorem together with Theorem 3.1(ii) tells us that

$$\begin{aligned}
&h_k\sum_{j=1}^{k}f\Big(z^k(t_j),\frac{z^k(t_j)-z^k(t_{j-1})}{h_k},t_j\Big)=\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}f(z^k(t_j),v^k(t),t_j)\,dt\\
&\sim\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}f(z^k(t_j),v^k(t),t)\,dt\sim\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}f(\bar x(t),v^k(t),t)\,dt\\
&=\int_0^T f(\bar x(t),v^k(t),t)\,dt\sim\int_0^T f(\bar x(t),\dot{\bar x}(t),t)\,dt,
\end{aligned}$$

where the sign '$\sim$' is used to indicate the equivalence as $k\to\infty$. Thus we get (3.32).

To proceed further, consider the numerical sequence

$$c_k:=\int_0^T|\dot{\bar x}^k(t)-\dot{\bar x}(t)|^2dt,\quad k\in I\!N,\tag{3.33}$$

and verify that $c_k\to 0$ as $k\to\infty$. Since the numerical sequence in (3.33) is obviously bounded, it has limiting points. Denote by $c\ge 0$ any of them and show that $c=0$. Arguing by contradiction, suppose that $c>0$. It follows from the uniform boundedness assumption (H1) and basic functional analysis that the sequence $\{\dot{\bar x}^k(\cdot)\}$ contains a subsequence (without relabeling), which converges in the weak topology of $L^2[0,T]$ to some $v(\cdot)\in L^2[0,T]$. Considering the absolutely continuous function

$$\widetilde{x}(t):=x_0+\int_0^t v(s)\,ds,\quad 0\le t\le T,$$

we deduce from the Newton-Leibniz formula that the sequence of the extended discrete trajectories $\bar{x}^k(\cdot)$ converges to $\widetilde{x}(\cdot)$ in the weak topology of $W^{1,2}[0,T]$, for which we have $\dot{\widetilde{x}}(t) = v(t)$ for a.e. $t \in [0,T]$. By invoking Mazur's weak closure theorem, it follows from the convexity of the sets $F(x,t)$ and the continuity of $F(\cdot,t)$ that the limiting function $\widetilde{x}(\cdot)$ satisfies the differential inclusion (3.2). Furthermore, the construction of the discrete approximation problems $(\widetilde{P}_k)$ with $\eta_k \to 0$ therein ensures that $\widetilde{x}(\cdot)$ is a feasible trajectory for the original Bolza problem $(\widetilde{P})$, and therefore for the relaxed (R) as well.

Employing again Mazur's weak closure theorem, we find a sequence of convex combination of $\dot{\bar{x}}^k(\cdot)$ converging to $\dot{\widetilde{x}}(\cdot)$ in the norm topology of $L^2[0,T]$ and hence a.e. on $[0,T]$ along some subsequence. Taking into account the construction of $\widehat{f}_F$ as the convexification of f_F in (2.6) with respect to the velocity variable, we arrive at the inequality

$$\int_0^T \widehat{f}_F(\widetilde{x}(t), \dot{\widetilde{x}}(t), t)\, dt \le \liminf_{k\to\infty} h_k \sum_{j=1}^{k} f\left(\bar{x}_j^k, \frac{\bar{x}_j^k - \bar{x}_{j-1}^k}{h_k}, t_j\right). \tag{3.34}$$

Define now the integral functional on $L^2[0,T]$ by

$$I[v] := \int_0^T |v(t) - \dot{\bar{x}}|^2 dt \tag{3.35}$$

and show it is convex on this space. Indeed, picking any $v(\cdot), w(\cdot) \in L^2[0,T]$ and $\lambda \in [0,1]$ and using the Cauchy-Schwartz inequality gives us

$$\begin{aligned} I[\lambda v + (1-\lambda)w] &= \int_0^T |\lambda(v(t) - \dot{\bar{x}}(t)) + (1-\lambda)(w(t) - \dot{\bar{x}}(t))|^2 dt \\ &\le \int_0^T \Big[\lambda|v(t) - \dot{\bar{x}}(t)| + (1-\lambda)|w(t) - \dot{\bar{x}}(t)|\Big]^2 dt \\ &= \lambda \int_0^T |v(t) - \dot{\bar{x}}(t)|^2 dt + (1-\lambda) \int_0^T |w(t) - \dot{\bar{x}}(t)|^2 dt \\ &= \lambda I[v] + (1-\lambda) I[w], \end{aligned}$$

which justifies the convexity and hence the lower semicontinuity of (3.35) in the weak topology of $L^2[0,T]$. It allows us to conclude that

$$\begin{aligned}\int_0^T |\dot{\widetilde{x}}(t) - \dot{\bar{x}}(t)|^2 dt &\le \liminf_{k\to\infty} \int_0^T |\dot{\bar{x}}^k(t) - \dot{\bar{x}}(t)|^2 dt \\ &= \liminf_{k\to\infty} \sum_{j=1}^k \int_{t_{j-1}}^{t_j} \Big| \frac{\bar{x}^k(t_j) - \bar{x}^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t) \Big|^2 dt.\end{aligned}$$

Employing this and passing to the limit in the constraints (3.25) and (3.26) for $\bar{x}^k(\cdot)$ yield

$$|\widetilde{x}(t) - \bar{x}(t)| \le \frac{\varepsilon}{2} \ \text{ for } \ t \in [0,T] \ \text{ and } \ \int_0^T |\dot{\widetilde{x}}(t) - \dot{\bar{x}}(t)|^2 dt \le \frac{\varepsilon}{2},$$

which verifies that the feasible trajectory $\widetilde{x}(\cdot)$ for (R) belongs to the prescribed $W^{1,2}[0,T]$ neighborhood of the r.i.l.m. $\bar{x}(\cdot)$ from the definition.

Now we are able to pass to the limit in the cost functional formula (3.24) in $(\widetilde{P}_k)$ for $\bar{x}^k(\cdot)$ by using (3.31), (3.34), and the assumption on $c_k \to c > 0$ in (3.33). It gives us

$$\widehat{J}[\widetilde{x}] = \varphi(\widetilde{x}(T)) + \int_0^T \widehat{f}_F(\widetilde{x}(t), \dot{\widetilde{x}}(t), t)\, dt \le \liminf_{k\to\infty} J_k[\bar{x}^k] + c < J[\bar{x}] = \widehat{J}[\bar{x}],$$

which contradicts the choice of $\bar{x}(\cdot)$ as a r.i.l.m. for the original Bolza problem $(\widetilde{P})$. Thus we have $c_k \to 0$ as $k \to \infty$ showing in this way that $\bar{x}^k(\cdot) \to \bar{x}(\cdot)$ strongly in $W^{1,2}[0,T]$.

To complete the proof of the theorem, it remains to justify the strong $C[0,T]$ convergence in (i) of discrete optimal trajectories for $(\check{P}_k)$ in the case when $\bar{x}(\cdot)$ is a strong local minimizers of the continuous-time Bolza problem $(\widetilde{P})$. Note that due to the convexity of $F(x,t)$ and the independence of the integrand f on the velocity variable, problem $(\widetilde{P})$ agrees with its relaxation (R). Taking into account the form of the cost functional (3.30) and Theorem 3.1(i) on the strong discrete approximation of $\bar{x}(\cdot)$ in $C[0,T]$, we arrive at the claimed convergence result in assertion (i) of this theorem by just simplifying the above proof of assertion (ii) and replacing the cost functional J_k with $\check{J}_k$. $\square$

3.4 Optimality Conditions for Discrete Approximations

In this section we derive necessary optimality conditions for each problem $(\widetilde{P}_k)$, $k \in I\!N$, in the sequence of discrete approximations formulated in Section 3.3. In the same way we can proceed with deriving necessary optimality conditions in the simplified problems $(\check{P}_k)$; we do not present them here due to the full similarity and size limitation.

Note that problems of this type intrinsically belong to nonsmooth optimization even when all the functions f and φ_i for $i = 0, \ldots, m+r$ are smooth and $\Omega = \mathbb{R}^n$. The nonsmoothness comes from the dynamic constraints in (3.9) given by the discretization of the differential inclusion (3.2); the number of these constraints is increasing along with decreasing the step of discretization. To derive necessary optimality conditions for problems $(\widetilde{P}_k)$, we employ advanced tools of variational analysis and generalized differentiation discussed in Chapter 2.

Now we employ Lemma 2.4 and calculus rules for generalized normals and subgradients to derive necessary optimality conditions for the structural dynamic problems of discrete approximation $(\widetilde{P}_k)$ in the extended Euler-Lagrange form. Note that for this purpose we need less assumptions that those imposed in (H1)–(H5). Observe also that the form of the Euler-Lagrange inclusion below reflects the essence of the implicit Euler scheme being significantly different from the adjoint system corresponding to the explicit Euler counterpart from [27, 30]. The solvability of the new implicit adjoint system is ensures by Lemma 2.4 due the given proof of the this theorem.

Theorem 3.3 (extended Euler-Lagrange conditions for discrete approximations). *Fix any $k \in I\!N$ and let $\bar{x}^k = (\bar{x}_0^k, \ldots, \bar{x}_k^k)$ with $\bar{x}_0^k = x_0$ in (3.9) be an optimal solution to problem $(\widetilde{P}_k)$ constructed in Section 3.3. Assume that the sets Ω and $\operatorname{gph} F_j$ with $F_j := F(\cdot, t_j)$ are closed and the functions φ_i for $i = 0, \ldots, m+r$ and $f_j := f(\cdot, \cdot, t_j)$ for $j = 0, \ldots, k$ are Lipschitz continuous around the corresponding points.*

Then there exist real numbers λ_i^k for $i = 0, \ldots, m+r$ and a vector $p^k := (p_0^k, \ldots, p_k^k) \in \mathbb{R}^{(k+1)n}$, which are not equal to zero simultaneously and satisfy the following relationships:

- *The sign conditions*

$$\lambda_i^k \geq 0 \ \text{ for } \ i = 0, \ldots, m; \tag{3.36}$$

- *the complementary slackness conditions*

$$\lambda_i^k [\varphi_i(\bar{x}_k^k) - L\eta_k] = 0 \ \text{ for } \ i = 1, \ldots, m; \tag{3.37}$$

- *the extended Euler-Lagrange inclusion held for $j = 1, \ldots, k$:*

$$\Big(\frac{p_j^k - p_{j-1}^k}{h_k}, p_{j-1}^k - \frac{\lambda_0^k \theta_j^k}{h_k}\Big) \in \lambda_0^k \partial f_j\Big(\bar{x}_j^k, \frac{\bar{x}_j^k - \bar{x}_{j-1}^k}{h_k}\Big) + N\Big(\Big(\bar{x}_j^k, \frac{\bar{x}_j^k - \bar{x}_{j-1}^k}{h_k}\Big); \operatorname{gph} F_j\Big) \tag{3.38}$$

- *the transversality inclusion*

$$-p_k^k \in \sum_{i=0}^{m} \lambda_i^k \partial \varphi_i(\bar{x}_k^k) + \sum_{i=m+1}^{m+r} \lambda_i^k \partial^0 \varphi_i(\bar{x}_k^k) + N(\bar{x}_k^k; \Omega_k), \tag{3.39}$$

where $\partial^0\varphi_i$ *stands for the symmetric subdifferential* (2.15) *of* φ_i*, and where*

$$\theta_j^k := -2\int_{t_j-1}^{t_j}\left(\dot{\bar{x}}(t) - \frac{\bar{x}_j^k - \bar{x}_{j-1}^k}{h_k}\right)dt. \tag{3.40}$$

Proof. Skipping for notational simplicity the upper index "k" if no confusions arise, consider the new "long" variable

$$z := (x_0, \dots, x_k, y_1, \dots, y_k) \in \mathbb{R}^{(2k+1)n} \ \text{ with the fixed initial vector } \ x_0$$

and for each $k \in I\!\!N$ reformulate the discrete approximation problem $(\widetilde{P}_k)$ as a mathematical program of the type (MP) with the following data:

$$\min \phi_0(z) := \varphi_0(x_k) + h_k\sum_{j=1}^{k} f(x_j, y_j, t_j) + \sum_{j=1}^{k}\int_{t_j-1}^{t_j}|y_j - \dot{\bar{x}}(t)|^2 dt \tag{3.41}$$

subject to the functional and geometric constraints

$$\phi_j(z) := |x_j - \bar{x}(t_j)|^2 - \frac{\varepsilon^2}{4} \le 0 \ \text{ for } \ j = 1, \dots, k, \tag{3.42}$$

$$\phi_{k+1}(z) := \sum_{j=1}^{k}\int_{t_j-1}^{t_j}|y_j - \dot{\bar{x}}(t)|^2 dt - \frac{\varepsilon}{2} \le 0, \tag{3.43}$$

$$\phi_{k+1+j}(z) = \varphi_j(x_k) - L\eta_k \le 0 \ \text{ for } \ j = 1, \dots, m+r, \tag{3.44}$$

$$\phi_{k+1+m+r+j}(z) := -\varphi_{m+j}(x_k) - L\eta_k \le 0 \ \text{ for } \ j = 1, \dots, r, \tag{3.45}$$

$$g_j(z) := x_j - x_{j-1} - h_k y_j = 0 \ \text{ for } \ j = 1, \ldots, k, \ \ g_0(z) = x(0) - x_0 \equiv 0, \tag{3.46}$$

$$z \in \Delta_0 = \{(x_0, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{(2k+1)n} | \ x_k \in \Omega\}, \tag{3.47}$$

$$z \in \Delta_j = \{(x_0, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{(2k+1)n} | \ y_j \in F_j(x_j)\}, \quad j = 1, \ldots, k. \tag{3.48}$$

Let $\bar{x}^k = (x_0, \bar{x}_1^k, \ldots, \bar{x}_k^k)$ be a given local optimal solution to problem $(\widetilde{P}_k)$, and thus the corresponding extended variable $\bar{z} := (x_0, \ldots, \bar{x}_k, (\bar{x}_1 - \bar{x}_0)/h_k, \ldots, (\bar{x}_k - \bar{x}_{k-1})/h_k)$, where the upper index "k" is omitted, gives a local minimum to the mathematical program (MP) with the data defined in (3.41)–(3.48). Applying now to $\bar{z}$ the generalized Lagrange multiplier rule from Lemma 2.4, we find normal collections

$$z_j^* = (x_{0j}^*, \ldots, x_{kj}^*, y_{1j}^*, \ldots, y_{kj}^*) \in N(\bar{z}; \Delta_j) \ \text{ for } \ j = 0, \ldots, k \tag{3.49}$$

and well as nonnegative multipliers $(\mu_0, \ldots, \mu_{k+1+m+2r})$ and vectors $\psi_j \in \mathbb{R}^n$ for $j = 0, \ldots, k$ such that we have the conditions

$$\mu_j \phi_j(\bar{z}) = 0 \ \text{ for } \ j = 1, \ldots, k + 1 + m + 2r, \tag{3.50}$$

$$- z_0^* - \ldots - z_k^* \in \partial \Big(\sum_{j=0}^{k+1+m+2r} \mu_j \phi_j \Big)(\bar{z}) + \sum_{j=0}^{k} (\nabla g_j(\bar{z}))^T \psi_j. \tag{3.51}$$

It follows from (3.49) and the structure of Δ_0 in (3.47) that

$$x_{k0}^* \in N(\bar{x}_k; \Omega_k), \ y_{00}^* = y_{i0}^* = x_{i0}^* = 0 \ \text{ for } i = 1, \ldots, k-1, \ \text{ and } \ x_{00}^* \ \text{ is free;}$$

the latter is due to the fact that x_0 is fixed. Furthermore, inclusion (3.49) for $j = 1, \ldots, k$ gives us by the structure of Δ_j that

$$(x^*_{jj}, y^*_{jj}) \in N\Big(\Big(\bar{x}_j, \frac{\bar{x}_j - \bar{x}_{j-1}}{h_k}\Big); \operatorname{gph} F_j\Big) \text{ and } x^*_{ij} = y^*_{ij} = 0 \text{ if } i \neq j,\ j = 1, \ldots, k.$$

Employing the above conditions together with the subdifferential sum rule from [29, Theorem 2.33] with taking into the nonnegativity of μ_j, we get from (3.51) that

$$\begin{aligned}
&\partial\Big(\sum_{j=0}^{k+1+m+2r} \mu_j\phi_j\Big)(\bar{z}) + \sum_{j=0}^{k}(\nabla g_j(\bar{z}))^T\psi_j \\
&\subset \sum_{j=0}^{k+1+m+2r} \mu_j\partial\phi_j(\bar{z}) + \sum_{j=0}^{k}(\nabla g_j(\bar{z}))^T\psi_j \\
&= \mu_0\nabla\Big[\varphi(x_k) + h_k\sum_{j=1}^{k} f(x_j, y_j, t_j) + \sum_{j=1}^{k}\int_{t_j-1}^{t_j}|y_j - \dot{\bar{x}}(t)|^2 dt\Big] \\
&\quad + \sum_{j=1}^{k}\mu_j\nabla(|x_j - \bar{x}(t_j)|^2) + \mu_{k+1}\nabla\Big(\sum_{j=1}^{k}\int_{t_j-1}^{t_j}|y_j - \dot{\bar{x}}(t)|^2 dt\Big) \\
&\quad + \sum_{j=1}^{m+r}\mu_{k+1+j}\nabla\varphi_i(\bar{x}_k) - \sum_{j=1}^{r}\mu_{k+1+m+r+j}\nabla\varphi_j(\bar{x}_k) \\
&\quad + \sum_{j=1}^{k}\nabla(x_j - x_{j-1} - h_k y_j)^T\psi_j + \nabla(x(0) - x_0)^T\psi_0,
\end{aligned}$$

where the derivatives (gradients, Jacobians) of all the *composite/sum* functions involves with respect of all their variables of are taken at the optimal point $\bar{z}$. It follows from Theorem 3.1 that for $k \in I\!N$ sufficiently large we have $\phi_j(\bar{z}^k) < 0$ for $\bar{z} = \bar{z}^k$ and $j = 1, \ldots, k+1$ due to the structures of the functions ϕ in (3.42) and (3.43) and the complementary slackness conditions in (3.50). This implies $\mu_j = 0$ for $j = 1, \ldots, k+1$. Considering now the Lagrange multipliers

$$\lambda^k_0 := \mu_0 \text{ and } \lambda^k_i := \mu_{k+1+i} \text{ for } i = 1, \ldots, m$$

and using the expressions for θ_j^k in (3.40), we find from the above subgradients

$$(v_j, w_j) \in \partial f_j(\bar{x}_j, \bar{y}_j),\ j = 1, \ldots, k, \quad u_i^k \in \partial\varphi_i(\bar{x}_k),\ i = 0, \ldots, m + r,$$

$$\text{and} \ \ u_i'^k \in \partial(-\varphi_i)(\bar{x}_k),\ i = m + 1, \ldots, m + r,$$

for which we have the conditions

$$-x_{jj}^* = \lambda_0^k h_k v_j + \psi_j - \psi_{j+1}, \quad j = 1, \ldots, k - 1,$$

$$-x_{k0}^* - x_{kk}^* = \lambda_0^k h_k v_k + \psi_k + \sum_{i=0}^{m} \lambda_0^k u_i^k + \sum_{i=m+1}^{m+r} \mu_{k+1+i} u_i^k + \sum_{i=m+1}^{m+r} \mu_{k+1+r+i} u_i'^k,$$

$$-y_{jj}^* = \lambda_0^k h_k w_j + \lambda_0^k \theta_j^k - h_k \psi_j, \quad j = 1, \ldots, k.$$

Next we introduce for each $k \in I\!N$ the adjoint discrete trajectories by

$$p_{j-1}^k := \psi_j^k \ \text{ for } \ j = 1, \ldots, k \quad \text{and}$$

$$p_k^k := -x_{k0}^* - \sum_{i=0}^{m} \lambda_i^k u_i^k - \sum_{i=m+1}^{m+r} \mu_{k+1+i} u_i^k - \sum_{i=m+1}^{m+r} \mu_{k+1+r+i} u_i'^k.$$

Then we get the relationships

$$\frac{p_j^k - p_{j-1}^k}{h_k} = \frac{\psi_{j+1}^k - \psi_j^k}{h_k} = \lambda_0^k v_j + \frac{x_{jj}^*}{h_k},$$

$$p_{j-1}^k - \frac{\lambda_0^k \theta_j^k}{h_k} = \psi_j^k - \frac{\lambda_0^k \theta_j^k}{h_k} = \lambda_0^k w_j + \frac{y_{jj}^*}{h_k},$$

which ensure the validity of the extended Euler-Lagrange inclusion of the theorem for each $j = 1, \ldots, k$. Furthermore, it follows from (3.44), (3.45) and the complementary slackness conditions in (3.50) that for, $j = m+1, \ldots, m+r$, we have

$$\mu_{k+1+j}(\varphi_j(x_k^k) - L\eta_k) = 0 \ \text{ and } \ \mu_{k+1+r+j}(-\varphi_j(x_k^k) - L\eta_k) = 0,$$

which implies that either $\mu_{k+1+j} = 0$ or $\mu_{k+1+r+j} = 0$ must be equal to zero for all $j = m+1, \ldots, m+r$. Denoting finally

$$\lambda_i^k := \begin{cases} \mu_{k+1+i} & \text{if } \ \mu_{k+1+r+i} = 0, \\ -\mu_{k+1+r+i} & \text{if } \ \mu_{k+1+i} = 0 \end{cases} \tag{3.52}$$

for each $i = m+1, \ldots, m+r$, we get

$$\begin{aligned} -p_k^k \ &= x_{k0}^* + \sum_{i=0}^{m} \lambda_i^k u_i^k + \sum_{i=m+1}^{m+r} \mu_{k+1+i} u_i^k + \sum_{i=m+1}^{m+r} \mu_{k+1+r+i} u_i'^k \\ &\in N(\bar{x}_k^k; \Omega_k) + \sum_{i=0}^{m} \lambda_i^k \partial\varphi_i(\bar{x}_k^k) \\ &\quad + \sum_{i=m+1}^{m+r} \mu_{k+1+i} \partial\varphi_i(\bar{x}_k^k) + \sum_{i=m+1}^{m+r} \mu_{k+1+r+i} \partial(-\varphi_i)(\bar{x}_k^k) \\ &\subset \sum_{i=0}^{m} \lambda_i^k \partial\varphi_i(\bar{x}_k^k) + \sum_{i=m+1}^{m+r} \lambda_i^k \partial^0 \varphi_i(\bar{x}_k^k) + N(\bar{x}_k^k; \Omega_k). \end{aligned}$$

This justifies the transversality inclusion completes the proof of the theorem. □

The last result of this section specifies the nontriviality condition of Theorem 3.3 (meaning that all the dual elements therein, i.e., λ_i^k for $i = 0, \ldots, m+r$ and p_j^k for $j = 0, \ldots, k$, are not equal to zero simultaneously) for the important class of multifunctions $F_j = F(\cdot, t_j)$ in the discrete inclusions (3.9) of the implicit Euler scheme satisfying the so-called Lipschitz-like (known also as Aubin's pseudo-Lipschitz) property around the optimal solution $\bar{x}^k$ for $(\widetilde{P}_k)$. Recall that a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *Lipschitz-like* around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ as well as a constant $\kappa \geq 0$ such that we have the inclusion

$$F(u) \cap V \subset F(x) + \kappa|x-u|I\!\!B \text{ for all } x,u \in U.$$

A crucial advantage of the nonconvex normal cone (2.11) is the possibility to obtain in its terms a complete characterization of the Lipschitz-like property of arbitrary closed-graph multifunctions. To formulate this result, we recall coderivative notion (2.16) in Chapter 2. Now we employ Lemma 2.3 to derive the aforementioned consequence of Theorem 3.3.

Corollary 3.4 (enhanced nontriviality condition). *In addition to the assumptions of Theorem* (3.3)*, suppose that for each* $j=1,\ldots,k$*, the multifunction* F_j *is Lipschitz-like around the optimal point* $(\bar{x}_j^k,(\bar{x}_j^k-\bar{x}_{j-1}^k)/h_k)$*. Then all the necessary optimality conditions of this theorem hold at* $\bar{x}^k$ *with the enhanced nontriviality*

$$\sum_{i=0}^{m+r}|\lambda_i^k|+|p_0^k|=1 \text{ for all } k\in I\!\!N. \tag{3.53}$$

Proof. If $\lambda_0^k=0$, then it follows from the Euler-Lagrange inclusion of the theorem that

$$\Big(\frac{p_j^k-p_{j-1}^k}{h_k},p_{j-1}^k\Big)\in N\Big(\Big(\bar{x}_j^k,\frac{\bar{x}_j^k-\bar{x}_{j-1}^k}{h_k}\Big);\operatorname{gph}F_j\Big)$$

for all $j=1,\ldots,k$, which tells us by the coderivative definition (2.16) that

$$\frac{p_j^k-p_{j-1}^k}{h_k}\in D^*F_j\Big(\bar{x}_j^k,\frac{\bar{x}_j^k-\bar{x}_{j-1}^k}{h_k}\Big)(-p_{j-1}^k),\quad j=1,\ldots,k.$$

Employing finally the coderivative criterion (2.17) with taking into account the transversality condition of the theorem as well as the normalization of $(\lambda_0, \ldots, \lambda_{m+r}, p_0^k)$ without changing other conditions, we arrive at (3.53) and thus completes the proof. □

3.5 Necessary Optimality Conditions for the Bolza Problem

In this section we come back to the generalized Bolza problem $(\widetilde{P})$ and prove necessary optimality conditions for an i.r.l.m. in the extended Euler-Lagrange form.

For the case of the explicit Euler discrete approximation, the necessary condition in a refined Euler-Lagrange form is obtained when the velocity function F is Lipschitzian around the local minimizer under consideration; cf. [27, Theorem 6.1] and [30, Theorem 6.22]. Similar, using the implicit Euler scheme and Theorem 3.3 the necessary condition for implicit Euler discrete approximation, we can get the Euler-Lagrange form for $(\widetilde{P})$ by passing to the limit with the vanishing step of dicretization with employing the coderivative calculations.

From the result of Section 3.4, here instead of the ROSL hypotheses in (H2) and the continuity hypotheses in (H4) and (H5), we need the Lipschitz continuity.

(H2') Given U from (H1), for all $x_1, x_2 \in U$, $t \in [0, T]$, we have

$$F(x_1, t) \subset F(x_2, t) + l_F|x_1 - x_2|I\!B. \tag{3.54}$$

(H4') $f(x, v, \cdot)$ is continuous for a.e. $t \in [0, T]$ and bounded uniformly in $(x, v) \in U \times (m_F B)$. There exists $\nu > 0$ and $l_f \geq 0$ such that the function $f(\cdot, \cdot, t)$ is locally Lipschitzian with modulus l_f around any point of the set $A_\nu(t)$ in (H4).

(H5') The function $\varphi_i(x(T))$ is Lipschitz continuous on U for all $i = 0, \ldots, m+r$ and Ω is locally closed around $x(T)$.

One of the fundamental properties of the generalized differential constructions under consideration is their *robustness* with respect to variables of differentiation. Actually, the subdifferential (2.14) turns out to be the *robust regularization* of the subdifferential mapping. It is well known that if f is locally Lipschitzian around $\bar{x}$ with modulus l_f, then $\partial f(\bar{x}) \neq \varnothing$, $|v| \leq l_f$ for all $v \in \partial f(\bar{x})$.

In the limiting procedure below, we also need the robustness of $\partial f(\cdot,\cdot,t)$ and $N((\cdot,\cdot);\operatorname{gph} F(\cdot,t))$

(H6) For a.e. $t \in [0,T]$ one has

$$\limsup_{(x',v')\to(\bar{x}(t),\dot{\bar{x}}(t)),t'\to t,t'<t} \partial f(x',v',t') = \partial f(\bar{x}(t),\dot{\bar{x}}(t),t).$$

(H7) For a.e. $t \in [0,T]$ one has

$$\limsup_{(x',v')\to(\bar{x}(t),\dot{\bar{x}}(t)),t'\to t,t'<t} N((x',v');\operatorname{gph} F(\cdot,t')) = N((\bar{x}(t),\dot{\bar{x}}(t));\operatorname{gph} F(\cdot,t)).$$

Note that the ROSL and Lipschitz-like properties of F used in Corollary 3.4 are generally independent (even for bounded mappings), and they both are implied by the classical Lipschitz condition.

Theorem 3.5 *Let $\bar{x}(\cdot)$ be an i.r.l.m. for problem $(\widetilde{P})$ under assumptions* (H1), (H2'), (H3'), (H4'), (H5'), (H6), *and* (H7). *Then there exist real numbers λ_i, $i = 0,\ldots,m+r$ and an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$, not both zero, such that for a.e. $t \in [0,T]$, the following necessary conditions hold:*

- *The sign conditions*

$$\lambda_i \geq 0 \;\; \textit{for} \;\; i = 0, \ldots, m;$$

- *the complementary slackness conditions*

$$\lambda_i \varphi_i(\bar{x}(T)) = 0 \;\; \textit{for} \;\; i = 1, \ldots, m;$$

- *the extended Euler-Lagrange inclusion*

$$\dot{p}(t) \in \text{clco}\Big\{ u | (u, p(t)) \in \lambda_0 \partial f(\bar{x}(t), \dot{\bar{x}}(t), t) + N((\bar{x}(t), \dot{\bar{x}}(t)); \text{gph}\, F(\cdot, t)) \Big\};$$

- *the transversality inclusion*

$$-p(T) \in \sum_{i=0}^{m} \lambda_i \partial \varphi_i(\bar{x}(T)) + \sum_{i=m+1}^{m+r} \lambda_i \partial^0 \varphi_i(\bar{x}(T)) + N(\bar{x}(T); \Omega).$$

Proof. Given the i.r.l.m. $\bar{x}(\cdot)$, since all the assumptions of Theorem 3.2 are satisfied, then the optimal solution $\bar{x}^k(t) = x^k(t_{j-1}) + (t - t_{j-1})(x^k(t_j) - x^k(t_{j-1}))/h_k$, for $t \in (t_{j-1}, t_j]$, $j = 1, \ldots, k$ to the discrete approximations (P_k) approximates $\bar{x}(t)$ in the norm of $W^{1,2}[0, T]$. Applying Theorem 3.3, we can find sequences of numbers $\lambda_i^k \geq 0$ and discrete adjoint trajectories $p^k = (p_1^k, \ldots, p_{k+1}^k)$ satisfied conditions (3.36)-(3.39).

By (3.53) from Corollary 3.4, we have for $i = 0, \cdots, m+r$, $|\lambda_i^k|$ and $|p^k|_0$ are bounded. Thus without loss of generality, we can suppose that $\lambda_i^k \to \lambda_i$ as $k \to \infty$ for all $i = 0, \ldots, m+r$. Thus the sign conditions is given from $\lambda_i^k \geq 0$ for $i = 0, \cdots, m$. Moreover, employing (3.20) $\eta_k \to 0$ as $k \to \infty$, we get the complementarity slackness conditions.

Consider the piecewise linear extensions of $p^k(t)$ on $[0,T]$,

$$p^k(t) := p^k_{j-1} + (t - t_{j-1})\frac{p^k_j - p^k_{j-1}}{h_k} \quad t \in (t_{j-1}, t_j],$$

with derivatives $\dot{p}^k(t) = (p^k_j - p^k_{j-1})/h_k$, $t \in (t_{j-1}, t_j]$. Having θ^k_j in Theorem 3.3, we consider a sequence of the functions

$$\theta^k(t) = \theta^k_j/h_k \ \text{ for } \ t \in (t_{j-1}, t_j], \quad j = 1, \ldots, k.$$

It from the strong convergence of Therorem 3.2

$$\begin{aligned} \int_0^T |\theta^k(t)|dt \quad &= \sum_{j=1}^k |\theta^k_j| \le 2\sum_{j=1}^k \int_{t_{j-1}}^{t_j} \Big|\dot{\bar{x}}(t) - \frac{\bar{x}^k_j - \bar{x}^k_{j-1}}{h_k}\Big|\, dt \\ &= 2\int_0^T |\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t)|\, dt \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{3.55}$$

Without loss of generality, suppose that

$$\dot{\bar{x}}^k(t) \to \dot{\bar{x}}(t) \quad \text{and} \quad \theta^k(t) \to 0 \ \text{ a.e. } \ t \in [0,T] \ \text{ as } \ k \to \infty. \tag{3.56}$$

Let us estimate the adjoint functions $p^k(\cdot)$ for large enough k. According to (3.38) and definition of the coderivative D^*F_j, there exist vectors $(v^k_j, w^k_j) \in \partial f_j(\bar{x}^k_j, (\bar{x}^k_j - \bar{x}^k_{j-1})/h_k)$ such that for all $j = 1, \ldots, k$ we have

$$\frac{p^k_j - p^k_{j-1}}{h_k} - \lambda^k_0 v^k_j \in D^*F_j\Big(\bar{x}^k_j, \frac{\bar{x}^k_j - \bar{x}^k_{j-1}}{h_k}\Big)(\lambda^k_0 w^k_j + \lambda^k_0\theta^k_j/h_k - p^k_{j-1}). \tag{3.57}$$

Since F is locally Lipschitzian, then by Lemma 2.3, for all $j = 1, \dots, k$, we have

$$\Big|\frac{p_j^k - p_{j-1}^k}{h_k} - \lambda_0^k v_j^k\Big| \le l_F|\lambda_0^k w_j^k + \lambda_0^k\theta_j^k/h_k - p_{j-1}^k|. \tag{3.58}$$

Employing the locally Lipschitzian of $f(\cdot,\cdot,t)$, by (H4') and $(v_j^k, w_j^k) \in \partial f_j(\bar{x}_j^k, (\bar{x}_j^k - \bar{x}_{j-1}^k)/h_k)$, we have for all $j = 1, \dots, k$ that $|(v_j^k, w_j^k)| \le l_f$, i.e.,

$$|v_j^k| \le l_f \text{ and } |w_j^k| \le l_f. \tag{3.59}$$

Employing (3.58)

$$|p_j^k| - |p_{j-1}^k| - h_k|\lambda_0^k v_j^k| \le h_k l_F\Big[|\lambda_0^k w_j^k| + |\lambda_0^k\theta_j^k/h_k| + |p_{j-1}^k|\Big],$$

using (3.53), (3.55), (3.59), for all $j = 1, \dots, k$, we get

$$\begin{aligned} |p_j^k| &\le (1 + h_k l_F)|p_j^k| + h_k l_f(1 + l_F) + l_F|\theta_j^k| \\ &\le (1 + h_k l_F)^j|p_0^k| + h_k l_f(1 + l_F)\Big[1 + (1 + h_k l_F) + \cdots + (1 + h_k l_F)^{j-1}\Big] \\ &\quad + l_F\Big[|\theta_j^k| + \cdots + (1 + h_k l_F)^{j-1}|\theta_1^k|\Big] \\ &\le e^{2l_F T}\Big[1 + l_f T(1 + l_F) + l_F 2\int_0^T |\theta k(t)|\, dt\Big] \end{aligned} \tag{3.60}$$

This means that the adjoint functions $p^k(t)$ are uniformly bounded in $[0, T]$. Employing (3.58), (3.59) to estimate the derivatives $\dot{p}^k(t)$, one has

$$|\dot{p}^k(t)| \le \Big|\frac{p_j^k - p_{j-1}^k}{h_k}\Big| \le l_f + l_F\big(l_f + |\theta^k(t)| + |p_{j-1}^k|\big), \quad t \in (t_{j-1}, t_j]. \tag{3.61}$$

By (3.56), (3.60), for all k large enough, $\dot{p}^k(t)$ are also uniformly bounded. Therefore, following Mordukhovich [27, 30], applied the Dunford theorem to the $\{\dot{p}^k(t)\}$ and by the weak continuity of integral, we can find an absolutely continuous function $p(\cdot)$ such that $p^k(\cdot) \to p(\cdot)$ uniformly in $[0,T]$ and $\dot{p}^k(\cdot) \to \dot{p}(\cdot)$ weakly in $L^1[0,T]$ for $k \to \infty$. Taking the limit in (3.53), one has the normalization condition $\sum_{i=0}^{m+r} |\lambda_i| + |p(0)| = 1$, which implies that $(\lambda_0, \dots, \lambda_{m+r})$ and $p(\cdot)$ are not equal to zero simultaneously. It follows from (3.60) that if $p(t_0) = 0$ at some point $t_0 \in [0,T]$, then $p(t) \equiv 0$ in $[0,T]$. Next, we conclude for $t \in (t_{j-1}, t_j]$, $j = 1, \dots, k$, the approximate Euler-Lagrange inclusion (3.38) can be rewritten as

$$\dot{p}^k(t) \in \Big\{ u \Big| \big(u, p^k(t_{j-1}) - \lambda_0^k \theta^k(t)\big) \in \lambda_0^k \partial f(\bar{x}_j^k, \dot{\bar{x}}^k(t), t_j) + N\big((\bar{x}_j^k, \dot{\bar{x}}^k(t)); \operatorname{gph} F(\cdot, t_j)\big) \Big\}. \quad (3.62)$$

According to the Mazur theorem, there is a sequence of convex combinations of $\dot{p}^k(t)$ that converges to $\dot{p}(t)$ for a.e. $t \in [0,T]$. Now passing to the limit in (3.62) as $k \to \infty$ and using (3.56) as well as hypotheses (H6) and (H7), we obtain the extended Euler-Lagrange inclusion.

Consider the approximate transversality inclusion, employing the local Lipschitz continuity of φ_i and the definition of subdifferential we have as $k \to \infty$

$$\sum_{i=0}^{m} \lambda_i^k \partial \varphi_i(\bar{x}_k^k) + \sum_{i=m+1}^{m+r} \lambda_i^k \partial^0 \varphi_i(\bar{x}_k^k) \to \sum_{i=0}^{m} \lambda_i \partial \varphi_i(\bar{x}(T)) + \sum_{i=m+1}^{m+r} \lambda_i \partial^0 \varphi_i(\bar{x}(T)).$$

Since $\Omega_k = \Omega + \eta_k I\!B$, passing to the limit $k \to \infty$, obviously we have the transversality inclusion and complete the proof of the theorem. $\triangle$

Chapter 4

Runge-Kutta Discrete Approximation of Nonconvex Differential Inclusions

4.1 Introduction

Consider the following optimization problem (P) of the *generalized Bolza type* governed by constrained differential inclusions:

$$\text{minimize } J[x] := \varphi\big(x(T)\big) + \int_0^T f\big(x(t), \dot{x}(t), t\big)dt \tag{4.1}$$

over absolutely continuous trajectories $x : [0, T] \to \mathbb{R}^n$ satisfying the differential inclusion

$$\dot{x}(t) \in F\big(x(t), t\big) \quad \text{a.e.} \quad t \in [0, T] \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n \tag{4.2}$$

subject to the geometric endpoint constraints

$$x(T) \in \Omega \subset \mathbb{R}^n, \tag{4.3}$$

Here x_0 is a fixed n-vector, $F\colon \mathbb{R}^n \times [0, T] \rightrightarrows \mathbb{R}^n$ is a set-valued mapping/multifunction, Ω is an nonempty set, f and φ are real-valued functions.

In this chapter we employ the Runge-Kutta method to solve the above Bolze problem under consideration allows us to build a well-posed sequence of optimization problems for discrete inclusions with a strong convergence of optimal solutions. Runge-Kutta methods first was used to solve optimal control problem with ordinary differential equations, not only second-order approximation was obtained but also the high-order approximations, see, e.g., [13, 15, 18, 19]. As the generalization of ordinary differential equations, differential inclusions problem also can be solved by Runge-Kutta method, see [1, 21, 23, 35, 36, 37] for more details. Since the special construction of set-valued mapping, in [35] Veliov got even for the different formalization of the second-order Runge-Kutta scheme, we may not suppose the same convergence rate. Instead of evaluating the convergence rate, we consider the possiblity that using the Runge-Kutta method to solve the optimization problem via an approximating Runge-Kutta sequence to the optimal solution. Consider the following formal generalization of second-order Runge-Kutta scheme

$$x(t+h) \in x(t) + 0.5h\Big\{y + F\big(x(t) + hy, t+h\big);\, y \in F\big(x(t), t\big)\Big\},\ h > 0.$$

The associate discrete inclusion is

$$x_{j+1} \in x_j + 0.5h\{y + F(x_j + hy, t_j + h);\ y \in F(x_j, t_j)\} \ \text{ with } \ h_k \downarrow 0 \ \text{ as } \ k \to \infty.$$

Our new results is outlined as follows. In section 4.2 we construct well-posted discrete Runge-Kutta approximations of differential inclusion. Under the assumption F is locally Lipschitzian we justify the strong approximation in the norm of $W^{1,2}[0,T]$ of a given feasible trajectory of the differential inclusion by discrete Rugn-Kutta trajectories.

Section 4.3 deals with discrete approximation of of Problem (P). We construct a sequence of optimization problems (P_k) to the original Bolza problem (P). Then we present the result on the strong stability of discrete approximations that justifies the $W^{1,2}$ norm convergence of optimal solutions for (P_k) to the given optimal solution $\bar{x}(\cdot)$ for (P).

Section 4.4 devoted to deriving necessary optimality conditions for the discrete approximation problems arising from the discrete approximation procedure whose well-posedness and stability are justified in Section 4.3. By Runge- Kutta discretization, the discrete problems can be regarded as a finite-dimensional mathematical program with a special structure. For different generalization of the parameter in the solution sequences for (P_k), we can get two version of necessary conditions.

4.2 Strong Approximation via Runge-Kutta Scheme

In this section, instead the one-sided Lipschitzian hypotheses in (**H2**), the locally Lipschitz continuity (**H2'**) is needed . We now construct a finite-difference approximation of the differential inclusion in (4.2) by using by the *Runge-Kutta method* , i.e. *Method of Euler-Cauchy* to replace the time derivative by

$$x(t+h) \in x(t) + 0.5h\Big\{y + F\big(x(t)+hy, t+h\big);\ y \in F(x(t),t)\Big\};\ \text{as}\ \ h \downarrow 0.$$

To formalize this process, for any $k \in I\!N$ define the discrete grid/mesh on $[0,T]$ by

$$T_k := \big(t_j \big|\ j = 0,1,\ldots,k\big)\ \text{ with }\ t_0 := 0,\ t_k := T,\ \text{ and stepsize }\ h_k := T/k = t_{j+1} - t_j.$$

Then the corresponding discrete inclusions associated with (4.2) via the Runge-Kutta scheme are constructed as follows:

$$x^k_{j+1} \in x^k_j + 0.5h_k\{y + F(x^k_j + h_k y, t_j + h);\ y \in F(x^k_j, t_j)\}\ \text{ for }\ j = 0,\ldots,k-1, \tag{4.4}$$

where the starting vector x^k_0 in (4.4) is taken from (4.2).

This method can be derived using two step:

$$\begin{cases} \bar{x}^k_{j+1} \in x^k_j + h_k F(x^k_j, t_j), \\ x^k_{j+1} \in (\bar{x}^k_{j+1} + x_j)/2 + 0.5 h_k F(\bar{x}^k_{j+1}, t_{j+1}), \end{cases}$$

The next theorem justifies the aforementioned strong $W^{1,2}[0,T]$-approximation of feasible solutions to (4.2) by those for the discrete inclusions (4.5).

Theorem 4.1 *Let $\bar{x}(\cdot)$ be a feasible trajectory for* (4.2) *under the assumptions in* (H1), (H2'), (H3)*. Then there is a sequence $\{z^k_j | j = 0, \ldots, k\}$ of feasible solutions to the discrete inclusions* (4.5) *such that the piecewise constantly extended to $[0,T]$ discrete velocity functions*

$$v^k(t) := \frac{z^k_{j+1} - z^k_j}{h_k}, \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, k-1, \tag{4.5}$$

converge to $\dot{\bar{x}}(\cdot)$ as $k \to \infty$ in the norm topology of $L^2[0,T]$, which is equivalent to the strong $W^{1,2}[0,T]$-convergence on $[0,T]$ of the piecewise linear functions constructed by

$$z^k(t) := x_0 + \int_0^t v^k(s)\, ds \ \ \textit{for all} \ \ t \in [0,T], \quad k = 1, 2, \ldots. \tag{4.6}$$

Proof. Without loss of generality, let $w^k(\cdot)$ be an arbitrary sequence of functions on $[0,T]$ such that $w^k(t)$ are constant on $[t_j, t_{j+1})$ and $w^k(t)$ converge to $\dot{\bar{x}}(t)$ as $k \to \infty$ in the norm topology of $L^1[0,T]$, taking into account the density of step functions in $L^1[0,T]$. Here $\bar{x}(t)$ is an arbitrary feasible trajectory for (4.2) from the formulation of the theorem and denote $\bar{x}_j := \bar{x}(t_j)$. By the boundness assumption in (H1), one gets

$$|w^k(t)| \le m_F + 1 \ \text{ for all } \ t \in [0,T] \ \text{ and } \ k \in I\!N.$$

Let us define further the sequences

$$w_j^k := w^k(t_j) \ \text{ for } \ j = 0, \ldots, k-1 \ \text{ and } \ \xi_k := \int_0^T |\dot{\bar{x}}(t) - w^k(t)|\, dt \to 0 \text{ as } k \to \infty \tag{4.7}$$

and for each $k \in I\!N$ form recurrently the collection of vectors $\{y_0^k, \ldots, y_k^k\}$ by

$$y_{j+1}^k := y_j^k + h_k w_j^k \ \text{ for } \ j = 0, \ldots, k-1 \ \text{ with } \ y_0^k = x_0. \tag{4.8}$$

Note that the continuous-time vector functions

$$y^k(t) := x_0 + \int_0^t w^k(s)\, ds, \quad 0 \le t \le T,$$

are piecewise linear extensions of the discrete ones (4.8) on $[0, T]$ satisfying the estimate

$$|y^k(t) - \bar{x}(t)| \le \int_0^t |w^k(s) - \dot{\bar{x}}(s)|\, ds \le \xi_k \ \text{ for all } \ t \in [0, T], \ k \in I\!N, \tag{4.9}$$

where ξ_k is taken from (4.7). Therefor, $y^k(t) \in U$ for all $t \in [0, T]$ if k is big enough; U as in $(H1)$. It well known that the Lipschitz condition (H2') is equivalent to

$$\text{dist}(w, F(x_1, t)) \le \text{dist}(w, F(x_2, t)) + l_F|x_1 - x_2|, \quad \forall w \in \mathbb{R}^n \ x_1, x_2 \in U, \ t \in [0, T].$$

For all $w, x \in \mathbb{R}^n$ and $t_1, t_2 \in [0, T]$, one obviously has

$$\text{dist}(w, F(x, t_1)) \le \text{dist}(w, F(x, t_2)) + \text{dist}(F(x, t_1), F(x, t_2)).$$

Since for $t \in [t_j, t_{j+1})$, one has $w^k(t) = w^k(t_j) = w_j$, for $j = 0, \ldots, k-1$, it follows from locally boundness and Lipschitzian of F and the construction of $y^k(t)$, for all $t \in [t_j, t_{j+1})$ we get

$$\begin{aligned}
\text{dist}(w_j^k, F(y_j^k, t))dt &= \text{dist}(w^k(t), F(y_j^k, t))dt \\
&\le \text{dist}(w^k(t), F(y^k(t), t))dt + l_F|y_j^k - y^k(t)| \\
&\le \text{dist}(w^k(t), F(x^k(t), t))dt + l_F|y^k(t) - x^k(t)| + l_F|y_j^k - y^k(t)| \\
&\le |w_j^k - \dot{\bar{x}}(t)| + l_F\xi_k + l_F(m_F + 1)(t - t_j).
\end{aligned}$$

This readily leads us to the following inequalities:

$$\begin{aligned}
&\sum_{j=1}^{k-1} \int_{t_j}^{t_j+1} \text{dist}(w_j^k, F(y_j^k, t))dt \\
&\le \sum_{j=1}^{k-1} \int_{t_j}^{t_j+1} \Big[|w_j^k - \dot{\bar{x}}(t)| + l_F\xi_k + l_F(m_F + 1)(t - t_j)\Big]\, dt \\
&\le \int_0^T |w^k(t) - \dot{\bar{x}}(t)|\, dt + Tl_F\xi_k + 0.5l_F(m_F + 1)h_kT \\
&\le \xi_k + Tl_F\xi_k + 0.5l_F(m_F + 1)h_kT.
\end{aligned} \tag{4.10}$$

Now we construct a sequence of discrete trajectories $z^k = (z_0^k, \dots, z_k^k)$ for (4.5) by the following *algorithmic procedure*:

$$v_{j1}^k \in F(z_j^k, t_j) \;\text{ with }\; |v_{j1}^k - w_j^k| = \text{dist}(w_j^k, F(z_j^k, t_j)),$$

$$v_{j2}^k \in F(z_j^k + h_k v_{j1}^k, t_{j+1}) \;\text{ with }\; |v_{j2}^k - w_j^k| = \text{dist}(w_j^k, F(z_j^k + h_k v_{j1}^k, t_{j+1})),$$

$$z_{j+1}^k = z_j^k + 0.5h_k(v_{j1}^k + v_{j2}^k),$$

One can directly see that z_{j+1}^k satisfies the Runge-Kutta scheme (4.4). Then

$$|z_{j+1}^k - y_{j+1}^k| \leq |z_j^k - y_j^k| + 0.5h_k \text{dist}(w_j^k, F(z_j^k, t_j)) + 0.5h_k \text{dist}(w_j^k, F(z_j^k + h_k v_{j1}^k, t_{j+1})).$$

Using the Lipschitz condition and the boundness assumption, we get

$$\text{dist}(w_j^k, F(z_j^k, t_j)) \leq l_F |z_j^k - y_j^k| + \text{dist}(w_j^k, F(y_j^k, t_j)),$$

and

$$\begin{aligned}
&\text{dist}(w_j^k, F(z_j^k + h_k v_{j1}^k, t_{j+1}))) \\
&\quad \leq \text{dist}(w_j^k, F(y_j^k, t_{j+1})) + l_F |z_j^k + h_k v_{j1}^k - y_j^k| \\
&\quad \leq \text{dist}(w_j^k, F(y^k, t_{j+1})) + l_F |z_j^k - y_j^k| + l_F h_k |v_{j1}^k - w_j^k + w_j^k| \\
&\quad \leq \text{dist}(w_j^k, F(y^k, t_{j+1})) + l_F |z_j^k - y_j^k| + l_F h_k \text{dist}(w_j^k, F(z^k, t_j)) + l_F h_k (m_F + 1) \\
&\quad \leq l_F h_k \text{dist}(w_j^k, F(y^k, t_j)) + \text{dist}(w_j^k, F(y^k, t_{j+1})) + l_F (1 + l_F h_k) |z_j^k - y_j^k| + l_F h_k (m_F + 1)
\end{aligned}$$

Thus we arrive at the estimate valid for all $j = 0, \ldots, k-1$ and $k \in I\!N$

$$\begin{aligned}
|z_{j+1}^k - y_{j+1}^k| &\leq (1 + l_F h_k + 0.5 l_F^2 h_k^2) |z_j^k - y_j^k| + 0.5 h_k (1 + l_F h_k) \text{dist}(w_j^k, F(y_j^k, t_j)) \\
&\quad + 0.5 h_k \text{dist}(w_j^k, F(y_j^k, t_{j+1})) + 0.5 h_k^2 l_F (m_F + 1) \\
&\leq (1 + l_F h_k)^2 |z_j^k - y_j^k| + 0.5 h_k (1 + l_F h_k) \text{dist}(w_j^k, F(y_j^k, t_j)) \\
&\quad + 0.5 h_k \text{dist}(w_j^k, F(y_j^k, t_{j+1})) + 0.5 h_k^2 l_F (m_F + 1).
\end{aligned} \tag{4.11}$$

Proceeding further by induction implies that

$$\begin{aligned}
&|z_{j+1}^k - y_{j+1}^k| \\
&\le 0.5h_k \sum_{m=0}^{j}(1+l_F h_k)^{2(j-m)}\Big[(1+l_F h_k)\mathrm{dist}(w_m^k, F(y_m^k, t_m)) \\
&\ \ +\mathrm{dist}(w_m^k, F(y_m^k, t_{m+1}))\Big] + 0.5h_k l_F m_F T \\
&\le 0.5e^{2l_F T}\textstyle\sum_{m=0}^{j} h_k\Big[(1+l_F h_k)\mathrm{dist}(w_m^k, F(y_m^k, t_m)) + \mathrm{dist}(w_m^k, F(y_m^k, t_{m+1}))\Big] + 0.5h_k l_F m_F T.
\end{aligned} \tag{4.12}$$

We recall next the average modulus of continuity of F defined by

$$\tau(F;h) := \sup{}_{x\in U}\int_0^T \sup\Big\{\mathrm{dist}(F(x,t'), F(x,t''))\Big|\ t', t'' \in \Big[t-\frac{h}{2}, t+\frac{h}{2}\Big]\Big\}\,dt$$

and consider the quantities ζ_k with the estimates

$$\begin{aligned}
\zeta_k\ &:= \sum_{m=1}^{k} h_k\Big[(1+l_F h_k)\mathrm{dist}(w_m^k, F(y_m^k, t_m)) + \mathrm{dist}(w_m^k, F(y_m^k, t_{m+1}))\Big] \\
&= \sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\Big[(1+l_F h_k)\mathrm{dist}(w_m^k, F(y_m^k, t_m)) + \mathrm{dist}(w_m^k, F(y_m^k, t_{m+1}))\Big] \\
&\le (2+l_F h_k)\sum_{m=1}^{k}\int_{t_{m-1}}^{t_m}\mathrm{dist}\big(w_m^k, F(y_m^k, t)\big) + (2+l_F h_k)\tau(F;h_k), \quad k \in I\!N.
\end{aligned} \tag{4.13}$$

It follows from (4.10) that

$$\zeta_k \le 2[\xi_k + Tl\xi_k + 0.5l_F(m_F+1)h_k T] + (2+l_F h_k)\tau(F;h_k).$$

It is well known (see, e.g., [30, Proposition 6.3]) that the a.e. continuity of $F(x,\cdot)$ on $[0,T]$ uniformly in $x \in U$ assumed in (H3) is equivalent to the convergence $\tau(F;h_k) \to 0$. By employing $\xi_k \to 0$ the definition of ζ_k, it gives us the convergence $\zeta_k \to 0$ as $k \to \infty$.

Using this and the last inequality in (4.12) allows us to conclude that for all $j = 0, \ldots, k-1$

and all $k \in I\!N$

$$|z_{j+1}^k - y_{j+1}^k| \leq 0.5\zeta_k e^{2l_F T} + 0.5h_k l_F m_F T. \tag{4.14}$$

Furthermore, we easily get the estimates

$$\begin{aligned} |z_{j+1}^k - \bar{x}_{j+1}| &\leq 0.5\zeta_k e^{2l_F T} + 0.5h_k l_F m_F T + |y_{j+1}^k - \bar{x}_{j+1}| \\ &\leq 0.5\zeta_k e^{2l_F T} + 0.5h_k l_F m_F T + \xi_k =: \eta_k, \end{aligned} \tag{4.15}$$

where $\eta_k \to 0$ due to (4.7) and $\zeta_k \to 0$ as $k \to \infty$.

Now let us estimate the quality $\int_0^T |\dot{z}^k(t) - \dot{x}^k(t)|\,dt$, considering next the the piecewise linear functions $z^k(\cdot)$ built in (4.6) by using the discrete velocity $v^k(\cdot)$ from (4.5), we get from (4.5) and (4.4) that

$$\dot{z}^k(t) = 0.5(v_{j1}^k + v_{j2}^k) = \frac{z_{j+1}^k - z_j^k}{h_k} \quad \text{on} \quad [t_j, t_j + 1), \quad j = 0, \ldots, k-1.$$

we derive from the construction of z_{j+1}^k and (4.14) that

$$\begin{aligned} \int_0^T |\dot{z}^k(t) - \dot{y}^k(t)|\,dt &= 0.5\sum_{j=0}^{k-1} h_k\Big[|v_{j1}^k - w_j^k| + |v_j^{k2} - w_j^k|\Big] \\ &\leq 0.5\sum_{j=0}^{k-1} h_k\Big[\text{dist}(w_j^k, F(z_j^k, t_j)) + \text{dist}(w_j^k, F(z_j^k + h_k v_{j1}^k, t_{j+1}))\Big] \\ &\leq 0.5\sum_{j=0}^{k-1} h_k\Big[|z_j^k - y_j^k| + 2\text{dist}(w_j^k, F(y_j^k, t_j)) + |z_j^k + h_k v_{j1}^k - y_j^k| + \tau(F; h)\Big] \\ &\leq 2T[0.5\zeta_k e^{2l_F T} + 0.5h_k l_F m_F T] + T h_k m_F + T\zeta_k. \end{aligned}$$

Since $\zeta_k \to 0$ as $k \to \infty$ for this sequence defined in (4.13), it follows from (4.14) that

$$\int_0^T |\dot{z}^k(t) - \dot{y}^k(t)|\,dt \to 0 \quad \text{as} \quad k \to \infty.$$

Employing further (4.9) gives us the estimate and the convergence

$$\alpha_k := \int_0^T |\dot{z}^k(t) - \dot{\bar{x}}(t)|\,dt \le \int_0^T |\dot{z}^k(t) - \dot{y}^k(t)|\,dt + \int_0^T |\dot{y}^k(t) - \dot{\bar{x}}(t)|\,dt \to 0, \tag{4.16}$$

which show that the sequence of $\dot{z}^k(\cdot)$ converge to $\dot{\bar{x}}(\cdot)$ in the norm topology of $L^1[0,T]$. To complete the proof, we need to verify $L^2[0,T]$ norm convergence. Similar with Theorem 3.1, by the constructions above and assumption (H1), it is implied by the following relationships:

$$\begin{aligned}
&\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left| \frac{z^k(t_{j+1}) - z^k(t_j)}{h_k} - \dot{\bar{x}}(t) \right|^2 dt \\
&\qquad = \sum_{j=0}^{k-1} \max\left(|v_j^k| + |\dot{\bar{x}}(t)| \right) \int_{t_j}^{t_{j+1}} |v_j^k - \dot{\bar{x}}(t)|\,dt \\
&\qquad \le 2m_F \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |v_j^k - \dot{\bar{x}}(t)|\,dt = 2m_F\alpha_k,
\end{aligned}$$

where α_k is taken from (4.16). This justifies the $W^{1,2}[0,T]$-norm convergence of the extended discrete trajectories $\{z^k(\cdot)\}$ from (4.6) in the first case under consideration.

△

4.3 Strong Convergence of Discrete Optimal Solutions

In this section we construct a sequence of optimization problems (P_k) for discrete inclusions (4.4) such that optimal solutions to (P_k) strongly (in $W^{1,2}[0,T]$ norm) converge to a given *r.i.l.m.* $\bar{x}(\cdot)$ in (P). Without loss of generality (due to (H1)) that $\alpha = 1$ and $p = 2$ in (2.5) and the definition of r.i.l.m.. Consider the problem (P) and $(\widetilde{P})$, the assumption (H5) in this chapter is as follows

(H5) The function φ is continuous on U and the set Ω is locally closed around $\bar{x}(T)$.

Fixed $\varepsilon > 0$ that is taken from (2.5) for the given r.i.l.m. $\bar{x}(\cdot)$. Take the sequence $\{\eta_k\}$ in (4.15) constructed via the approximation of the local optimal solution $\bar{x}(\cdot)$ under consideration. Then we define a sequence of discrete approximation problems (P_k), $k \in I\!N$, as follows:

$$\begin{aligned}\text{minimize } J_k[x^k] : \ &= \varphi_0(x^k(t_k)) + h_k \sum_{j=0}^{k-1} f\Big(x^k(t_j), \frac{x^k(t_{j+1}) - x^k(t_j)}{h_k}, t_j\Big) \\ &+ \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big|\frac{x^k(t_{j+1}) - x^k(t_j)}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt\end{aligned} \tag{4.17}$$

over trajectories $x^k = (x_0^k, \ldots, x_k^k)$ of the discrete inclusions (4.4) subject to the constraints

$$|x^k(t_j) - \bar{x}(t_j)|^2 \le \frac{\varepsilon^2}{4} \quad \text{for} \ \ j = 1, \ldots, k, \tag{4.18}$$

$$\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big|\frac{x^k(t_{j+1}) - x^k(t_j)}{h_k} - \dot{\bar{x}}(t)\Big|^2 dt \le \frac{\varepsilon}{2}, \tag{4.19}$$

$$x_k^k \in \Omega_k := \Omega + \eta_k I\!B. \tag{4.20}$$

The next theorem shows that the discrete optimal solutions of problems (P_k) converge to a given r.i.l.m. $\bar{x}(\cdot)$ of (P).

Theorem 4.2 *Let $\bar{x}(\cdot)$ be an i.r.l.m. for the original Bolza problem (P) under the validity of assumptions* (H1), (H2'), (H3)-(H5) *around $\bar{x}(\cdot)$. Then any sequence $\bar{x}^k(\cdot)$ whenever $k \in I\!N$ of optimal solutions to problem (P_k) piecewise linearly extended to $[0, T]$ converges to $\bar{x}(\cdot)$ as $k \to \infty$ in the norm topology of $W^{1,2}[0, T]$.*

Proof. Similar as Theorem 3.2, we prove that for each k big enough, the discrete trajectory $\{z_j^k\}$ constructed in Theorem 4.1 is a feasible solution to (P_k). Next we proceed with the strong approximation for any sequence of the discrete optimal solutions to (P_k) piecewise linearly extended to $[0, T]$. Let us prove for any sequence of optimal solutions $\bar{x}^k$ to (P_k) has

$$\liminf_{k\to\infty} J_k[\bar{x}^k] \leq J[\bar{x}]. \tag{4.21}$$

To accomplish this, it suffices to prove

$$\lim{}_{k\to\infty} J_k[z^k] = J[\bar{x}],$$

Since $\bar{x}^k$ is optimal solution, one can gets for each k, $J_k[\bar{x}^k] \leq J_k[z^k]$.

Consider expression (4.17) for $J_k[z^k]$, due to continuity of φ and the strong appromation of $\{z_j^k\}$ in Theorem 4.1, we have as $k \to \infty$,

$$\varphi(z^k(t_k)) \to \varphi(\bar{x}(T)),$$

$$\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \| \frac{z_k(t_{j+1}) - z_k(t_j)}{h_k} - \dot{\bar{x}}(t)\|^2 dt \to 0.$$

Employing now Lebesgue's dominated convergence theorem together with Theorem 4.1 tells us that

$$\begin{aligned}
&h_k \sum_{j=0}^{k-1} f(z_k(t_j), \frac{z_k(t_{j+1}) - z_k(t_j)}{h_k}, t_j) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(z_k(t_j), v^k(t), t_j)\, dt \\
&\sim \sum_{j=1}^{k} \int_{t_j}^{t_{j+1}} f(z_k(t_j), v^k(t), t)\, dt \sim \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f(\bar{x}(t), v^k(t), t) \\
&= \int_0^T f(\bar{x}(t), v^k(t), t)\, dt \sim \int_0^T f(\bar{x}(t), \dot{\bar{x}}(t), t)\, dt.
\end{aligned}$$

The last $\sim$ is by Theorem 3.2, $v^k \to \dot{\bar{x}}$ pointwise a.e. on [0,T], by bounded convergence Lebesgue Theorem we can get $J_k[z^k] \to J[\bar{x}]$, which implies (4.21) .

To proceed further, consider the numerical sequence

$$c_k := \int_0^T |\dot{\bar{x}}^k(t) - \dot{\bar{x}}(t)|^2 dt, \quad k \in I\!N, \tag{4.22}$$

and verify that $c_k \to 0$ as $k \to \infty$. Since the numerical sequence in (4.22) is obviously bounded, it has limiting points. Denote by $c \geq 0$ any of them and show that $c = 0$. Arguing by contradiction, suppose that $c > 0$. It follows from the uniform boundedness assumption (H1) and basic functional analysis that the sequence $\{\dot{\bar{x}}^k(\cdot)\}$ contains a subsequence (without relabeling), which converges in the weak topology of $L^2[0,T]$ to some $v(\cdot) \in L^2[0,T]$. Considering the absolutely continuous function

$$\widetilde{x}(t) := x_0 + \int_0^t v(s)\,ds, \quad 0 \leq t \leq T,$$

we deduce from the Newton-Leibniz formula that the sequence of the extended discrete trajectories $\bar{x}^k(\cdot)$ converges to $\widetilde{x}(\cdot)$ in the weak topology of $W^{1,2}[0,T]$, for which we have $\dot{\widetilde{x}}(t) = v(t)$ for a.e. $t \in [0,T]$. According to the Mazur's weak closure theorem, there is a sequence of convex combinations of $\bar{x}^k(\cdot$ that converges to $\widetilde{x}(\cdot)$ in the norm topology of $L^2[0,T]$, Hence it contains a subsequence converging to $\widetilde{x}(\cdot)$ for a.e. $t \in [0,T]$. Follows from the continuity of $F(\cdot,t)$ that $\widetilde{x}(\cdot)$ is a feasible trajectory for the relaxed (R). Taking into account the construction of $\widehat{f}_F$ as the convexification of f_F in (2.6) with respect to the velocity variable, we arrive at the inequality

$$\int_0^T \widehat{f}_F(\widetilde{x}(t), \dot{\widetilde{x}}(t), t)\,dt \leq \liminf_{k\to\infty} h_k \sum_{j=1}^k f\Big(\bar{x}_j^k, \frac{\bar{x}_j^k - \bar{x}_{j-1}^k}{h_k}, t_j\Big). \tag{4.23}$$

Observe that the integral functional

$$I[v] := \int_0^T |v(t) - \dot{\bar{x}}|^2 dt$$

is convex and lower semicontinuous in weak topology of $L^2[0,T]$.

It allows us to conclude that

$$\begin{aligned}\int_0^T |\dot{\widetilde{x}}(t) - \dot{\bar{x}}(t)|^2 dt &\le \liminf_{k\to\infty} \int_0^T |\dot{\bar{x}}^k(t) - \dot{\bar{x}}(t)|^2 dt \\ &= \liminf_{k\to\infty} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left| \frac{\bar{x}^k(t_j) - \bar{x}^k(t_{j-1})}{h_k} - \dot{\bar{x}}(t) \right|^2 dt.\end{aligned}$$

Now passing to the limit in the constraints (4.18) and (4.19) for $\bar{x}^k(\cdot)$ yield $\widetilde{x}(\cdot)$ belongs to the prescribed $W^{1,2}[0,T]$ neighborhood of the r.i.l.m. $\bar{x}(\cdot)$ from the definition.

Now we are able to pass to the limit in the cost functional formula (4.17) in (P_k) for $\bar{x}^k(\cdot)$ by using (4.21), (4.23), and the assumption on $c_k \to c > 0$ in (4.22). It gives us

$$\widehat{J}[\widetilde{x}] = \varphi(\widetilde{x}(T)) + \int_0^T \widehat{f}_F(\widetilde{x}(t), \dot{\widetilde{x}}(t), t)\, dt \le \liminf_{k\to\infty} J_k[\bar{x}^k] + c < J[\bar{x}] = \widehat{J}[\bar{x}],$$

which contradicts the choice of $\bar{x}(\cdot)$ as a r.i.l.m. for the original Bolza problem (P). Therefore, one has $c = 0$ which establishes $\bar{x}^k(\cdot) \to \bar{x}(\cdot)$ strongly in $W^{1,2}[0,T]$.

△

4.4 Optimality Conditions for Discrete Approximations

In this section we derive necessary optimality conditions for each problem (P_k), $k \in I\!N$, in the sequence of discrete approximations formulated in Section 4.1. To derive necessary optimality conditions for problems (P_k), we employ advanced tools of variational analysis and generalized differentiation discussed in Chapter 2.

Since the Runge-Kutta method can be derived using two step, the calculation of the sum of two set value mapping needed. We first prove the following Corollary.

Corollary 4.3 *Let* $F\colon \mathbb{R}^n \times [0,T] \to \mathcal{CC}(\mathbb{R}^n)$, $\Omega_1 = \{(\bar{x}, v_1, v_2) | v_1 \in F(\bar{x}, t_0)\}$, *and* $\Omega_2 = \{(\bar{x}, v_1, v_2) | v_2 \in F(\bar{x} + hv_1, t_0 + h)\}$, *Then*

$$N((\bar{x}, v_1, v_2), \Omega_1 \cap \Omega_2) \subset N((\bar{x}, v_1, v_2), \Omega_1) + N((\bar{x}, v_1, v_2), \Omega_1).$$

Proof. Following from Lemma 2.2, we only need to prove the normal qualification condition holds for Ω_1, Ω_2.

Pick up arbitrary $(x^*, -v_1^*, -v_2^*) \in N((\bar{x}, v_1, v_2), \Omega_1) \cap N((\bar{x}, v_1, v_2), \Omega_1)$, by the definition of Ω_1, we have $v_2^* = 0$ and $x^* \in D^*F(\bar{x}, v_1)(v_1^*)$.

Let$g(\bar{x}, v_1) = \bar{x}, v_1$, define $G(\bar{x}, v_1) = F(g(\bar{x}, v_1), t_0 + h)$. Obviously g is strictly differentiable, then $D^*G(\bar{x}, v_1, v_2) = \nabla g(\bar{x}, v_1)^* D^*F(g(\bar{x}, v_1), v_2)$, where $v_2 \in G(\bar{x}, v_1)$. By the definition of Ω_2, then $(x^*, -v_1^*) \in D^*G(\bar{x}, v_1, v_2)(v_2^*) = (1, h)y^* = (y^*, hy^*)$, where $y^* \in D^*F(g(\bar{x}, v_1), v_2)(v_2^*)$. Now using the coderivative property of the Lipschitz continuous function, we have $y^* = 0$ from $v_2^* = 0$. Then $x^* = 0$, $v_1^* = 0$. Thus the normal qualification condition holds. We proved $N((\bar{x}, v_1, v_2), \Omega_1 \cap \Omega_2) \subset N((\bar{x}, v_1, v_2), \Omega_1) + N((\bar{x}, v_1, v_2), \Omega_1)$.

△

Now we employ Lemma 2.4 and calculus rules for generalized normals and subgradients to derive necessary optimality conditions for the structural dynamic problems of discrete approximation (P_k) in the extended Euler-Lagrange form. Note that for this purpose we need less assumptions that those imposed in (H1)–(H5). Observe also that the form of the Euler-Lagrange inclusion below reflects the essence of the implicit Euler scheme being significantly different from the adjoint system corresponding to the explicit Euler counterpart from [27, 30]. The solvability of the new implicit adjoint system is ensures by Lemma 2.4 due the give n proof of the this theorem.

Theorem 4.4 (necessary conditions for Runge-Kutta approximations I). *Fix any* $k \in I\!N$ *and let* $\bar{x}^k = (\bar{x}_0^k, \ldots, \bar{x}_k^k)$ *with* $\bar{x}_0^k = x_0$ *in* (4.2) *be an optimal solution to problem* (P_k) *constructed in Section* 4.3. *Assume that the sets* Ω *and* $\mathrm{gph}\, F_j$ *with* $F_j := F(\cdot, t_j)$ *are closed and the function* φ *and* $f_j := f(\cdot,\cdot,t_j)$ *for* $j = 0,\ldots,k$ *are Lipschitz continuous around the corresponding points. Then there exist real numbers* λ^k *and a vector* $p^k := (p_0^k, \ldots, p_k^k) \in \mathbb{R}^{(k+1)n}$, *which are not equal to zero, such that*

$$\begin{aligned}&\Big(\frac{p_{j+1}^k - p_j^k}{h_k}, 0.5p_j^k - 0.5\frac{\lambda_0^k\theta_j^k}{h_k}, 0.5p_j^k - 0.5\frac{\lambda_0^k\theta_j^k}{h_k}\Big)\\ &\in \Big(\lambda_0^k\partial f_j\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big)_1, 0.5\lambda_0^k\partial f_j\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big)_2, 0.5\lambda_0^k\partial f_j\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big)_2\Big)\\ &\quad + N\Big(\Big(\bar{x}_j^k, v_{j1}, v_{j2}\Big); \Delta_{j1}\Big) + N\Big(\Big(\bar{x}_j^k, v_{j1}, v_{j2}\Big); \Delta_{j2}\Big); \;\; \text{for } j = 0,\ldots,k-1;\end{aligned}$$

$$-p_k^k \in \lambda^k\partial\varphi(\bar{x}_k^k) + N(\bar{x}_k^k, \Omega_k),$$

where

$$\Delta_{j1} = \big\{(\bar{x}_j^k, v_{j1}, v_{j2}) \,\big|\, v_{j1} \in F_j(\bar{x}_j^k)\big\}, \tag{4.24}$$

$$\Delta_{j2} = \big\{(\bar{x}_j^k, v_{j1}, v_{j2}) \,\big|\, v_{j2} \in F_{j+1}(\bar{x}_j^k + h_k v_{j1})\big\}, \tag{4.25}$$

$$\theta_j^k := -\int_{t_j}^{t_{j+1}} \Big(\dot{\bar{x}}(t) - \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big)\, dt. \tag{4.26}$$

Proof. Skipping for notational simplicity the upper index "k" if no confusions arise, consider the new "long" variable

$$z := (x_0, \ldots, x_k, v_{01}, \ldots, v_{k-1\,1}, v_{02}, \ldots, v_{k-1\,2}) \in \mathbb{R}^{(3k+1)n} \;\; \text{with the fixed initial vector } \; x_0$$

and for each $k \in I\!N$ reformulate the discrete approximation problem (P_k) as a mathematical program of the type (MP) in Chapter 2 with the following data:

$$\min \phi(z) := \varphi_0(x_k) + h_k \sum_{j=0}^{k-1} f(x_j, \frac{v_{j1}+v_{j2}}{2}, t_j) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} |\frac{v_{j1}+v_{j2}}{2} - \dot{\bar{x}}(t)|^2 dt \tag{4.27}$$

subject to the functional and geometric constraints

$$\phi_j(z) := |x_j - \bar{x}(t_j)|^2 - \frac{\varepsilon^2}{4} \le 0 \ \text{ for } \ j = 1, \ldots, k, \tag{4.28}$$

$$\phi_{k+1}(z) := \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} |\frac{v_{j1}+v_{j2}}{2} - \dot{\bar{x}}(t)|^2 dt - \frac{\varepsilon}{2} \le 0, \tag{4.29}$$

$$g_j(z) := x_{j+1} - x_j - h_k \frac{v_{j1}+v_{j2}}{2} = 0 \ \text{ for } \ j = 0, \ldots, k-1, \ \ g_k(z) = x(0) - x_0 \equiv 0, \tag{4.30}$$

$$\begin{aligned} z \in \Delta_j = \{ \ & (x_0, \ldots, x_k, v_{01}, \ldots, v_{k-1\,1}, v_{02}, \ldots, v_{k-1\,2}) \in \mathbb{R}^{(3k+1)n} \\ & \big| \ v_{j1} \in F_j(x_j), \ v_{j2} \in F_{j+1}(x_j + h_k v_{j1}) \}, \quad j = 0, \ldots, k-1, \end{aligned} \tag{4.31}$$

$$z \in \Delta_k = \{(x_0, \ldots, x_k, v_{01}, \ldots, v_{k-1\,1}, v_{02}, \ldots, v_{k-1\,2}) \in \mathbb{R}^{(3k+1)n} \big| \ x_k \in \Omega_k\}. \tag{4.32}$$

Let $\bar{x}^k = (x_0, \bar{x}_1^k, \ldots, \bar{x}_k^k)$ be a given local optimal solution to problem (P_k), and thus by the Runge-Kutta scheme (4.4), there exist $v_{j,1}^k \in F_j(x_j^k)$ such that

$$\bar{x}_{j+1}^k - \bar{x}_j^k \in 0.5h\{v_{j1}^k + F_{j+1}(x_j^k + h_k v_{j1}^k)\}.$$

Let $v_{j2}^k = 2(\bar{x}_{j+1}^k - \bar{x}_j^k)/h_k - v_{j1}^k$, we get the corresponding extended variable

$$\bar{z} := (x_0, \ldots, \bar{x}_k, v_{01}, \ldots, v_{k-1\,1}, v_{02}, \ldots, v_{k-1\,2})$$

, where the upper index "k" is omitted, gives a local minimum to the mathematical program

(MP) with the data defined in (4.27)–(4.32). Applying now to $\bar{z}$ the generalized Lagrange multiplier rule from Lemma 2.4, for $j = 0, \ldots, k$ we find normal collections

$$z_j^* = (x_{0j}^*, \ldots, x_{kj}^*, v_{0j,1}^*, \ldots, v_{k-1j,1}^*, v_{0j,2}^*, \ldots, v_{k-1j,2}^*) \in N(\bar{z}; \Delta_j) \tag{4.33}$$

and well as nonnegative multipliers $(\mu_0, \ldots, \mu_{k+1})$ and vectors $\psi_j \in \mathbb{R}^n$ for $j = 0, \ldots, k$ such that we have the conditions

$$\mu_j \phi_j(\bar{z}) = 0 \ \text{ for } \ j = 1, \ldots, k+1, \tag{4.34}$$

$$- z_0^* - \ldots - z_k^* \in \partial\Big(\sum_{j=0}^{k+1} \mu_j \phi_j\Big)(\bar{z}) + \sum_{j=0}^{k} (\nabla g_j(\bar{z}))^T \psi_j. \tag{4.35}$$

It follows from (4.33) for $j = 0, \ldots, k$ gives us by the structure of Δ_j that

$$(x_{jj}^*, v_{jj,1}^*, v_{jj,2}^*) \in N\Big((\bar{x}_j, v_{j1}, v_{j2}); \Delta_j\Big) \ \text{ and } \ x_{ij}^* = v_{ij,1}^* = v_{ij,2}^* = 0 \text{ if } \ i \neq j, \ j = 0, \ldots, k-1.$$

$$x_{kk}^* \in N\big(\bar{x}_k; \Omega_k\big)$$

Employing the above conditions together with the subdifferential sum rule from [29, Theorem 2.33] with taking into the nonnegativity of μ_j, we get from (4.35) that

$$\begin{aligned}
&\partial\Big(\sum_{j=0}^{k+1} \mu_j \phi_j\Big)(\bar{z}) + \sum_{j=0}^{k} (\nabla g_j(\bar{z}))^T \psi_j \subset \sum_{j=0}^{k+1} \mu_j \partial \phi_j(\bar{z}) + \sum_{j=0}^{k} (\nabla g_j(\bar{z}))^T \psi_j \\
&= \mu_0 \nabla \Big[\varphi(x_k) + h_k \sum_{j=0}^{k-1} f\Big(x_j, \frac{v_{j1} + v_{j2}}{2}, t_j\Big) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big|\frac{v_{j1} + v_{j2}}{2} - \dot{\bar{x}}(t)\Big|^2 dt\Big] \\
&\ + \sum_{j=1}^{k} \mu_j \nabla(|x_j - \bar{x}(t_j)|^2) + \mu_{k+1} \nabla\Big(\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big|\frac{v_{j1} + v_{j2}}{2} - \dot{\bar{x}}(t)\Big|^2 dt\Big) \\
&\ + \sum_{j=0}^{k-1} \nabla(x_{j+1} - x_j - 0.5 h_k (v_{j1} + v_{j2}))^T \psi_j + \nabla(x(0) - x_0)^T \psi_k,
\end{aligned}$$

where the derivatives (gradients, Jacobians) of all the *composite/sum* functions involves with respect of all their variables of are taken at the optimal point $\bar{z}$.

Considering now the Lagrange multipliers $\lambda^k := \mu_0$, and using the expressions for θ_j^k in (4.26), we find from the above subgradients

$$u_0 \in \partial\varphi(\bar{x}_k), \quad (v_j, w_j) \in \partial f_j(\bar{x}_j, \bar{y}_j),\ j = 0, \ldots, k-1$$

for which we have the conditions

$$-x_{00}^* = \lambda^k h_k v_0 - \psi_0 - \psi_k, ,$$

$$-x_{jj}^* = \lambda^k h_k v_j + \psi_j - \psi_{j+1}, \quad j = 1, \ldots, k-1,$$

$$-x_{kk}^* = \lambda^k u_0 + \psi_{k-1}, \quad j = 1, \ldots, k-1,$$

$$-v_{jj} := -v_{jj,1}^* = 0.5\lambda_0^k h_k w_j + \lambda_0^k \theta_j^k - 0.5 h_k \psi_j = -v_{jj,2}^*, \quad j = 0, \ldots, k-1.$$

Next we introduce for each $k \in I\!N$ the adjoint discrete trajectories by

$$p_{j+1}^k := \psi_j^k \ \text{ for } \ j = 0, \ldots, k-1,$$

$$p_0^k = \psi_k^k \quad \text{and} \quad p_k^k := -x_{kk}^* + \lambda^k u_0.$$

Then for each $j = 0, \ldots, k-1$, we get the relationships

$$\frac{p_{j+1}^k - p_j^k}{h_k} = \frac{\psi_{j+1}^k - \psi_j^k}{h_k} = \lambda_0^k v_j + \frac{x_{jj}^*}{h_k},$$

$$p_{j+1}^k - \frac{\lambda_0^k \theta_j^k}{h_k} = \psi_{j+1}^k - \frac{\lambda_0^k \theta_j^k}{h_k} = \lambda_0^k w_j + \frac{2v_{jj}^*}{h_k}.$$

$p_k^k := -x_{kk}^* + \lambda^k u_0$ given us the transversality inclusion $-p_k^k \in \lambda^k \partial\varphi(\bar{x}_k^k) + N(\bar{x}_k^k, \Omega_k)$. Employing Corollary 4.3, where $\Delta_j = \Omega_{j1} \cap \Omega_{j2}$, we have

$$N((\bar{x}_j, v_{j1}, v_{j2}), \Delta_j) \subset N((\bar{x}_j, v_{j1}, v_{j2}), \Omega_{j1}) + N((\bar{x}_j, v_{j1}, v_{j2}), \Omega_{j2})$$

which ensure the validity of the necessary conditions of the theorem for each $j = 0, \ldots, k-1$. $\triangle$

If consider other formalization of Δ_j(4.31) for $j = 0, \ldots, k-1$ as

$$\Delta_j = \Big\{(x_0, \ldots, x_k, v_0, \ldots, v_{k-1}) \in \mathbb{R}^{(2k+1)n} \Big|\ v_j \in 0.5\{y + F_{j+1}(x_j + h_k y)\};\ y \in F_j(x_j)\Big\}, \quad , \tag{4.36}$$

then we can find another version of necessary conditions for the discrete problems.

Theorem 4.5 (necessary conditions for Runge-Kutta approximations II). *Fix any $k \in I\!N$ and let $\bar{x}^k = (\bar{x}_0^k, \ldots, \bar{x}_k^k)$ with $\bar{x}_0^k = x_0$ in (4.2) be an optimal solution to problem (P_k) constructed in Section 4.3. Assume that the sets Ω and $\operatorname{gph} F_j$ with $F_j := F(\cdot, t_j)$ are closed and the function φ and $f_j := f(\cdot, \cdot, t_j)$ for $j = 0, \ldots, k$ are Lipschitz continuous around the corresponding points.*

Then there exist real numbers λ^k and a vector $p^k := (p_0^k, \ldots, p_k^k) \in \mathbb{R}^{(k+1)n}$, which are not equal to zero, such that for $j = 0, \ldots, k-1$

$$\Big(\frac{p_{j+1}^k - p_j^k}{h_k}, p_{j+1}^k - \frac{\lambda_0^k \theta_j^k}{h_k}\Big) \in \lambda_0^k \partial f_j\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big) + N\Big(\Big(\bar{x}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big); \operatorname{gph} G_j\Big),$$

$$-p_k^k \in \lambda^k \partial\varphi_i(\bar{x}_k^k) + N(\bar{x}_k^k, \Omega_k),$$

where

$$G_j(\bar{x}_j^k) := \Big\{0.5y + 0.5F_{j+1}(\bar{x}_j^k) \mid y \in F_j(\bar{x}_j^k)\Big\}, \tag{4.37}$$

$$\theta_j^k := -\int_{t_j}^{t_{j+1}} \Big(\dot{\bar{x}}(t) - \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}\Big)\, dt. \tag{4.38}$$

Proof. Similar as Theorem 4.4, now let the new "long" variable z as

$$z := (x_0, \ldots, x_k, v_1, \ldots, v_{k-1}) \in \mathbb{R}^{(2k+1)n} \ \text{ with the fixed initial vector } \ x_0$$

and for each $k \in I\!N$ reformulate the discrete approximation problem (P_k) as a mathematical program with the following data:

$$\min \phi(z) := \varphi_0(x_k) + h_k \sum_{j=0}^{k-1} f(x_j, v_j, t_j) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} |v_j - \dot{\bar{x}}(t)|^2 dt$$

subject to the functional and geometric constraints

$$\phi_j(z) := |x_j - \bar{x}(t_j)|^2 - \frac{\varepsilon^2}{4} \le 0 \ \text{ for } \ j = 1, \ldots, k,$$

$$\phi_{k+1}(z) := \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} |v_j - \dot{\bar{x}}(t)|^2 dt - \frac{\varepsilon}{2} \le 0,$$

$$g_j(z) := x_{j+1} - x_j - h_k v_j = 0 \ \text{ for } \ j = 0, \ldots, k-1, \ \ g_k(z) = x(0) - x_0 \equiv 0,$$

$$z \in \Delta_j = \Big\{ \ (x_0, \ldots, x_k, v_0, \ldots, v_{k-1}) \in \mathbb{R}^{(2k+1)n}$$
$$\Big| \ v_j \in 0.5\{y + F_{j+1}(x_j + h_k y)\};\ y \in F_j(x_j)\Big\}, \quad j = 0, \ldots, k-1,$$

$$z \in \Delta_k = \Big\{(x_0, \dots, x_k, v_1, \dots, v_{k-1}) \in \mathbb{R}^{(2k+1)n} \Big| \; x_k \in \Omega_k\Big\}.$$

Repeat the some steps as in the proof of Theorem 4.4, the proof of this theorem can be completed. △

A natural question arises about the relationship between these two version. To do this, we need to analysis the relationship between $N\Big((\bar{x}_j, v_{j1}, v_{j2}); \Delta_j\Big)$ and $N\Big(\Big(\bar{x}_j, \frac{v_{j1}+v_{j2}}{2}\Big); \Delta_j\Big)$. This is an open question of our further research. Actually, comparing the necessary conditions in Theorem 3.3 and Theorem 4.5, we will find these two theorem are similar. the only difference is since these conditions are based on different schemes, the G_j in Theorem 4.5 replace the F_j in Theorem 3.3.

Chapter 5

Discussion

This dissertation develops constructive numerical approach to investigate the generalized Bolza problem of optimizing constrained differential inclusions by using two discrete approximations methods: the implicit Euler scheme and the Runge-Kutta scheme. In this way we not only justify the well-posedness of the suggested discrete approximation procedures in the sense of either the uniform or $W^{1,2}$-convergence of discrete optimal solutions to a given local (strong or intermediate) minimizer of the original nonsmooth Bolza problem, but also derive necessary optimality conditions to solve each problem of the corresponding discrete approximations. As mentioned in the introductory Chapter 1, the results obtained are new even in case of the numerical method even for unconstrained differential inclusions satisfying the classical Lipschitz condition.

In Section 3.5, we get the necessary optimality conditions for the given intermediate local minimizer of the original problem (P) under the condition that the velocity function F is Lipschitzian. A natural question arises about the possibility to derive the necessary optimality conditions for the given intermediate or strong local minimizer of the original problem (P) for favorable classes of ROSL differential inclusions by passing to the limit from those obtained for the implicit Euler discrete approximations $(\widetilde{P}_k)$ and the Runge-Kutta discrete approximations (P_k), respectively, as $k \to \infty$.

On the other hand, the method of discrete approximations has been successfully employed in [7] to derive necessary optimality conditions for the Bolza problem governed by a dissipative (hence ROSL while unbounded and heavily non-Lipschitzian) differential inclusion that arises in optimal control of Moreau's *sweeping process* with mechanical applications. The procedure in [7] exploits some specific features of the controlled sweeping process over convex polyhedral sets, and thus a principal issue of the our further research is about the possibility to extend these results to more general ROSL differential inclusions.

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ABSTRACT

FINITE-DIFFERENCE APPROXIMATIONS AND OPTIMAL CONTROL OF DIFFERENTIAL INCLUSIONS

by

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Advisor: Prof. Boris S. Mordukhovich

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

This dissertation concerns the study of the generalized Bolza type problem for dynamic systems governed by constrained differential inclusions. We develop finite-discrete approximations of differential inclusions by using the implicit Euler scheme and the Runge-Kutta scheme for approximating time derivatives, while an appropriate well-posedness of each approximation is justified. We establish the uniform approximation of strong local minimizers for the continuous-time Bolza problem by optimal solutions to the implicitly discretized finite-difference systems in the general ROSL setting and even by the strengthen W1;2-norm approximation of this type in the case "intermediate" (between strong and weak minimizers) local minimizers under additional assumptions. We derive the strong approximation of feasible trajectories for the Lipschitzian differential inclusions and also the strong convergence of optimal solutions to the corresponding dynamic optimization problems under Runge-Kutta discrete approximations. Finally, we derive necessary optimality conditions for each scheme for the discretized Bolza problems via suitable generalized differential constructions of variational analysis.

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List of Publications and Preprints

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B.S. Mordukhovich and T.Yuan, Runge-Kutta discrete approximations and optimization of differential inclusions, in preparation